

Circuit Analysis

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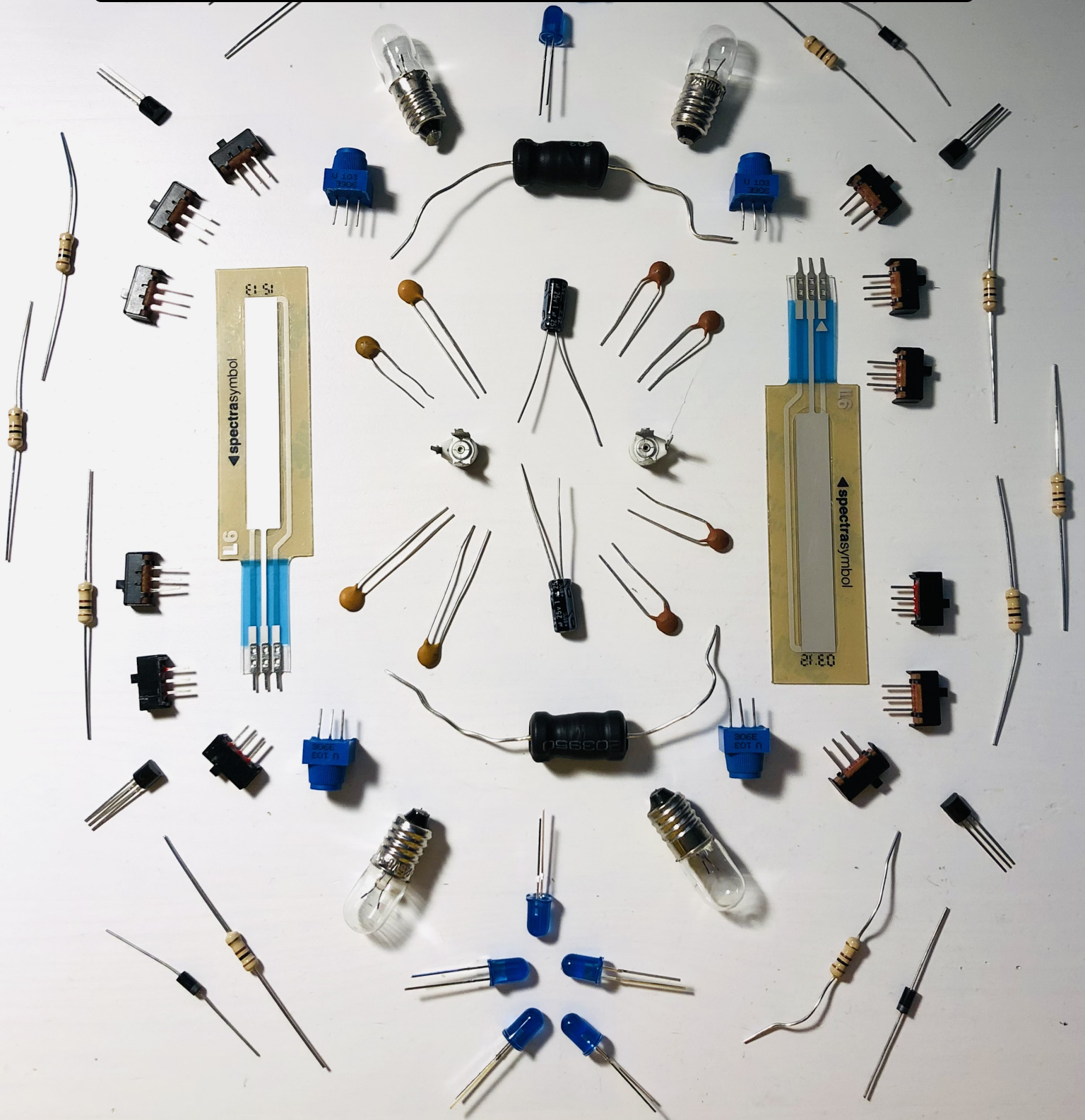


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Author Note

This book is an attempt to compile together my notes for Circuit Analysis (ENGIN-2210 at the College of DuPage). I want students to have a free resource that they can use instead of paying hundreds of dollars for a book that they may not read and does not align exactly with my course content.

That said, this book will always be a work in progress. I cannot guarantee that it is free from typos. I will, however, do my best to implement your feedback if you find any issues with the text. Feel free to e-mail me at pasqualea185@cod.edu with your notes.

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The entirety of this work was created by Alyssa J. Pasquale, Ph.D. The cover photograph is by the author and is a collection of circuit components (capacitors, trimmer pots, trimmer capacitors, soft potentiometers, inductors, toggle switches, transistors, resistors, diodes, light-emitting diodes, and incandescent lamps). All circuit diagrams, equations, and figures in this text were created by the author using L^AT_EX libraries.

Symbol Notation

A note about symbols used in this text: DC quantities will be represented by capital letters such as V for voltage, I for current, and P for power. Time-varying quantities will be represented by lowercase letters such as $v(t)$ for voltage, $i(t)$ for current, and $p(t)$ for power.

Changelog

This table will be edited when changes are made to the textbook.

Date	Chapter(s)	Description of Change(s)
2021-09-14	all	First edition of this book published online
2021-10-24	solutions	Corrected coefficients A_1 and A_2 in chapter 7 question 3 Corrected typo in chapter 7 question 10 (solution is underdamped) Corrected phase angle of $v(t)$ in chapter 9 question 9 Corrected phase angle of $v(t)$ in chapter 9 question 14 Rewrote questions 21 and 22 in chapter 9
2022-03-08	2	Rewrote supermesh steps for mesh analysis
2022-03-09	4	Corrected cascading inverted op-amp gain equation and derivation
2022-03-21	3	Corrected typo in chapter 3 source transformation example
	5	Added an explanation of dielectric permittivity
	6	Added a separate equation for magnetic permeability
2022-04-07	7	Corrected minor typos
2022-05-05	all	Added chapter and section names to all page headers
	10	Defined trigonometric identity used in instantaneous power derivation
	10	Defined the * symbol (complex conjugate)
	7	Corrected captions on non-homogeneous example problems
	8	Corrected typo (sin had been written as cos) in negative amplitude sine example
2022-05-11	contents	Corrected formatting issue
2022-05-19	1	Corrected typo in calculating current from ramp function example equation
2022-06-03	11	Added a chapter on Laplace transforms and s-domain analysis
2022-07-08	11	Corrected circuit diagram of s-domain transform of inductor
2022-07-11	11	Corrected typo in second derivative property in properties of the Laplace transform table
2023-02-02	1, 4	Corrected minor grammatical mistakes
2023-02-27	3	Corrected minor grammatical mistakes
2023-04-12	5–11	Specified $v(0^+)$ vs. $v(0^-)$ where appropriate Specified $i(0^+)$ vs. $i(0^-)$ where appropriate Included $u(-t)$ terms in equations, where appropriate
	all	Corrected formatting inconsistencies
2023-05-05	10	Corrected typo in correcting a capacitive load equation (S_{LOAD} had used Z_{EQ} instead of Z_{LOAD} , but the result was correct)
2023-05-13	10	Corrected minor grammatical mistakes
2023-05-18	11	Corrected typo in first order circuit s-domain analysis example Corrected typo in partial fraction expansion with complex conjugate roots example
2024-04-12	7	Corrected typo in homogeneous general second order circuit in symbolic form example
2024-04-18	9	Corrected typo in band-pass filter analysis example

1 Properties of Electric Circuits

Electric circuits have many different properties. Some of these properties describe quantities that we are interested in measuring in a circuit. Properties such as voltage and current are things that engineers are interested in designing circuits around; we want circuits to have particular values of voltage or current at specific points in a design. It's important for engineers to have a solid understanding of these properties. They will form the basis for the remainder of the content of this textbook.

1.1 Charge

Charge is a fundamental property of matter, just as mass is a fundamental property of matter. Charge is important because it interacts through electromagnetic forces. By physically separating two or more charges, an electric field is generated that creates either attractive or repulsive forces. These forces can direct the movement of charges throughout a circuit. Charge comes from protons and electrons. Protons each contain one unit of positive charge (usually denoted as $+e$), which is equal to 1.6×10^{-19} C. C stands for coulomb, which is the SI unit used for charge. Electrons each contain one unit of negative charge, which is equal to -1.6×10^{-19} C. The symbol used for charge is Q .

1.2 Voltage

Voltage, also referred to as electric potential, potential difference, or electromotive force, is another important property of circuits. Voltage defines how much work is done per unit of charge to move a “test charge” between two points. While voltage can be an abstract property to understand, it is one of the three most important properties of electric circuits (they are: voltage, current, and resistance). The symbol used for voltage is V and the units are V (volts).

A useful way to understand voltage can be to compare it with its gravitational equivalent. While gravitational forces and electromagnetic forces have many differences, they also have many similarities. Gravitational potential energy describes the amount of energy that an object has by virtue of its mass and its distance away from a reference mass. However, if that object were to disappear, the reference mass would still exist, and that reference mass has the *potential* to generate energy if an object of mass were to find itself some distance (r) away from the reference mass. This is depicted schematically in figure 1.1. This *potential* to have energy is known as gravitational potential.

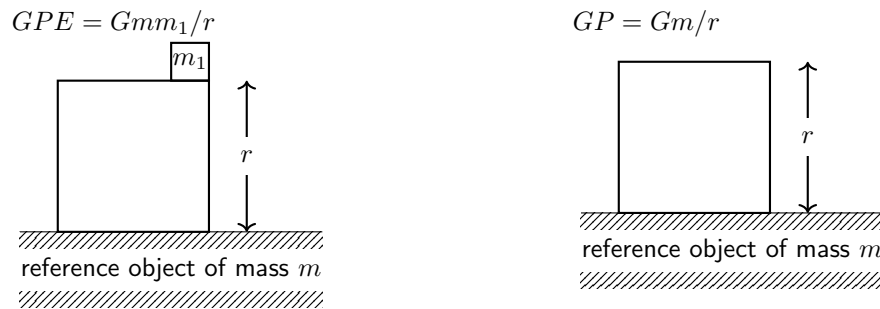


Figure 1.1: Schematic depiction of gravitational potential energy and gravitational potential.

As humans living on the planet Earth, gravitational potential depends on the mass of the Earth, the radius of the Earth, and the universal gravitational constant ($G = 6.7 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$). While gravitational potential energy is a concept that you have probably studied extensively in your physics classes, you have probably not spent as much time on gravitational potential.

Electric potential energy describes the energy that an object has by virtue of its charge and its distance away from a reference charge. However, if that object were to disappear, the reference charge would still exist, and that reference charge has the *potential* to generate energy if an object of charge were to find itself some distance (r) away from the reference charge. This is depicted schematically in figure 1.2. This *potential* to have energy is known as electric potential, or voltage.



Figure 1.2: Schematic depiction of electric potential energy and electric potential (voltage).

Voltage therefore depends on the charge of the reference object, the distance between the two objects, as well as Coulomb's constant ($k = 8.99 \times 10^9 \text{ Nm}^2/\text{C}^2$).

What is important in the context of circuit analysis is where voltage comes from and how it will be treated as a quantity. Voltage can be generated in several different ways. Static electric fields are generated by physically separating charges and keeping them apart from each other, which is what a battery accomplishes using chemical means. Voltage can also be created by a time-varying magnetic field or charges moving through a static magnetic field. These methods are how generators and power plants work.

In an electromagnetic physics class, the emphasis is generally on using Maxwell's equations to solve for voltage as a quantity. In electrical engineering, we will treat voltage as either a source that is applied to a circuit (by means of a power supply, batteries, or function generator) and as a quantity to be measured over various circuit elements.

1.2.1 Voltage as a Difference

It is important to reiterate that voltage is a potential *difference*. This implies that voltage must be measured with respect to two points in a circuit. The terminology used in this book is to mention the “voltage dropped over an element.” Voltage is also *relative*, which means that care needs to be taken in the measurement or calculation of voltage drops. In other words, measuring the voltage drop with voltmeter leads in one orientation will lead to the negative value of the measurement with the leads in the other orientation. (In mathematical terms, $V_{ab} = -V_{ba}$.) Because voltage is relative, it is important to keep in mind what the reference is when understanding what is meant by a positive or a negative voltage drop.

It is also important to discuss the concept of a reference potential. Generally, we use ground as a reference potential of zero volts. The symbol for ground that is used on a circuit diagram resembles an upside-down tree and is depicted in figure 1.3.



Figure 1.3: Circuit symbol used for ground.

The Earth serves as a good choice of zero potential (ground) in circuits. The Earth can absorb very large amounts of charge without experiencing a change in this potential. Equipment with Earth ground connections will typically have a copper wire that is buried into the Earth in order to facilitate this ability to transfer charge.

Some objects (such as cars, airplanes, and some consumer electronic devices with portable power sources) do not have a physical connection to Earth ground, but still make use of a reference voltage of zero potential. For example, an airplane will typically use the metal chassis as a ground. However, this can lead to static buildup as charge cannot always dissipate from the metal chassis. When an airplane is refueled, this can be a particular hazard. For this reason, a fuel truck will ground itself to the airplane chassis to reduce any possibility of static buildup and discharge. Electrostatic discharge can also be a hazard when using integrated circuits, particularly those used for assembling computers. Care and consideration must be taken to ensure that people are properly grounded before and during working with these objects.

1.2.2 Measuring Voltage

Voltage can be measured in a circuit using a device called a voltmeter. (A device that can measure multiple properties – such as voltage, current, and resistance – is called a multimeter.) The circuit symbol for a voltmeter is shown in figure 1.4. A voltmeter contains two leads: a positive and a reference (or common) lead. When placed across the elements of a circuit, the potential difference between the two points (potential at positive lead minus the potential at common lead) is measured. Modern digital voltmeters typically use analog to digital converters and microcontrollers to turn a voltage measurement into a digital readout. Note that it is important to establish the reference potential and ensure that the reference lead is in the correct

location. Otherwise the voltmeter will display the negative of the desired value. ($V_{ab} = -V_{ba}$)

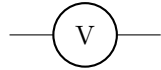


Figure 1.4: Circuit symbol used for a voltmeter.

Due to Kirchhoff's voltage law (explained in section 2.7.2 in this book), voltages must be measured in parallel. In other words, if the voltage drop over a particular circuit element is to be measured, then one lead of the voltmeter should be on one end of the element with the other lead of the voltmeter on the other end of the element. This is shown graphically in figure 1.5.

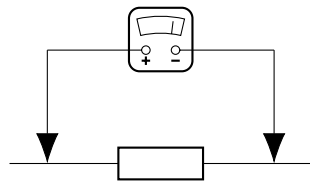


Figure 1.5: A diagram of how voltage is measured in parallel (across a component).

It is important for a measuring tool to provide minimal impact on the functionality of a circuit. Because a voltmeter is placed in parallel with a circuit element, it is important for current flow into the voltmeter to be minimal. If current were to flow into the voltmeter, then it would change the current flowing through the circuit and would impact the measurement of the voltage. For this reason, voltmeters are designed with a very high internal resistance. This makes voltmeters rather safe to use, as placing a voltmeter incorrectly cannot lead to an accidental low-resistance short circuit.

1.3 Current

Current quantifies the amount of charge moving past a point per unit of time (Q/t); if charge changes with time, then current is equal to the time derivative of charge, depicted in equation 1.1. The symbol for current is I .

$$i(t) = \frac{dq(t)}{dt} \quad (1.1)$$

Because the units of charge are coulombs and the unit of time are seconds, the unit for current is C/s, also known as A (ampere, or amp).

Negatively charged particles will move through a conductor in the presence of an electric field to an area of high potential (positive charge). Current is defined opposite to the motion of negative charges. In other words: current is defined opposite to the motion of electrons in a circuit.

1.3.1 Current as a Flow

Current relates to the flow of charge through an object. Because the charge is moving in a particular direction, the sign of current correlates to the direction of the current. It is possible to calculate or measure current in any arbitrary direction. If the calculation or measurement yields a positive result, it means that the direction that was chosen is indeed the direction in which current is flowing. If the calculation or measurement yields a negative result, it means that current is flowing in the opposite direction.

As an example, let's assume that we have a circuit element and we believe that current is traveling from left to right, as shown in figure 1.6. If we measure or calculate the current and obtain a positive result, it means that the current is indeed flowing from left to right. However, if our measurement or calculation leads to a negative result, it means that the current is flowing from right to left.

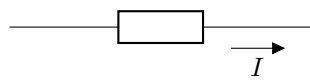


Figure 1.6: A circuit element; we are assuming that current flows from left to right.

1.3.2 Measuring Current

Current can be measured in a circuit using a device called an ammeter. The circuit schematic for an ammeter is shown in figure 1.7. An ammeter contains two leads. Current enters through one lead and exits through the reference (common) lead. When placed in series with a circuit element, the current flowing through the ammeter is measured. An ammeter will measure the current flowing from one lead to the common lead; therefore it is important to establish which connection to make with the common lead to ensure that the readout is not going to be the negative value of what is desired.

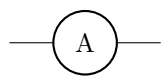


Figure 1.7: Circuit symbol used for an ammeter.

Modern digital ammeters use a resistor with a very small value (called a shunt resistor) over which the voltage can be measured using an analog to digital converter. Using Ohm's law, the current can be calculated using a microcontroller and displayed on a screen.

Due to Kirchhoff's current law (explained in section 2.7.1 in this book), currents must be measured in series. In other words, the path of the circuit must be broken open, and the ammeter must be inserted into that path in order to measure the current flowing through that path. This is shown graphically in figure 1.8.

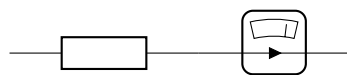


Figure 1.8: A diagram of how current is measured in series (in the same path as a component).

Just as with a voltmeter, it is important for an ammeter to disturb the functionality of the circuit as little as possible. An ideal ammeter would contribute no voltage drop to the circuit. However, as a shunt resistor is required to measure current, that means the shunt resistor should have as small a value as possible. Because the internal resistance of an ammeter is so low, it is imperative to use caution when connecting an ammeter. If a lot of current is caused to flow through an ammeter (by connecting the ammeter in a path with very low resistance, for example by connecting it in parallel with an element rather than in series with an element), then the internal circuitry can be damaged or a fuse blown.

1.4 Power

Energy is lost or gained at particular rates in each element in a circuit. This rate of energy loss (or gain) is known as power. The symbol for power is P , and the units are W (watts). In general, power is defined as the time derivative of energy or work in a circuit element ($dw(t)/dt$). The instantaneous power consumed by a circuit element is given in equation 1.2.

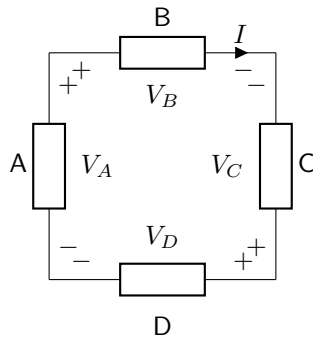
$$p(t) = i(t)v(t) \tag{1.2}$$

1.4.1 Conservation of Power

Conservation of power, which comes from conservation of energy, is an important concept in electric circuits. This tells us that **all power that is created or delivered in a circuit is equal to all of the power that is absorbed or consumed in that circuit.** Some circuit elements create or deliver power. Other circuit elements absorb power. To determine which thing (delivering or absorbing) a circuit element is doing, there are two methods to use. The first by being told what the circuit element is. Batteries, power supplies, and function generators deliver power. Resistors absorb power.

The second method is by analyzing the voltage drop over and current flow through a circuit element. If current flows from high potential to low potential in a circuit element, the element absorbs power. If current flows from low potential to high potential in a circuit element, the circuit delivers power.

Example: Determining if circuit elements deliver or absorb power



Current flows from high to low potential in elements B and D: therefore B and D are absorbing power.
 Current flows from low to high potential in elements A and C: therefore A and C are delivering power.

1.5 Power Sources

Power sources are used in a circuit to generate energy and distribute it to other circuit elements. The types of power supplies that are typically used in circuits include batteries, constant voltage sources, constant current sources, and function generators.

1.5.1 Direct Current (DC) Sources

Some sources are direct current (DC), which means that the value of the current or voltage supplied by the source does not change with time. Three circuit symbols for DC power sources are shown in figure 1.9.

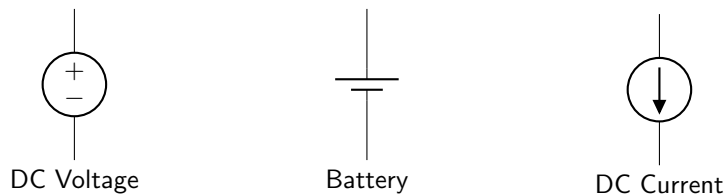


Figure 1.9: Three types of DC power sources: a voltage source, battery, and current source.

The DC voltage power supply contains plus and minus symbols. The plus sign represents the side of the supply that corresponds to high potential. The minus sign represents the side of the supply that corresponds to low potential. The battery does not contain plus or minus signs but instead has two lines: one long and one short. The long end (which is the upper line in figure 1.9) corresponds to high potential, and the short end corresponds to low potential. In the symbol for a current source, the arrow depicts the direction of current flow (assuming that the source is properly connected in a complete circuit).

1.5.2 Alternating Current (AC) Sources

In alternating current (AC) sources, the voltage or current supplied by the source is time-varying. Four circuit symbols for AC power supplies are shown in figure 1.10.

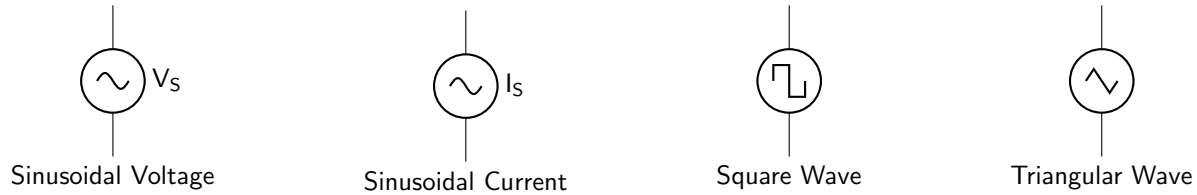


Figure 1.10: Four types of AC power sources: sinusoidal voltage, sinusoidal current, square wave, and triangular wave.

The key to analyzing AC power supply circuit symbols is to look at the labels on the sources. The symbol within the circle depicts the nature of the time-varying source: sinusoidal, square wave, or triangular wave. The label or units on the label (given in V, A, or mA) depicts if the source is a voltage source or current source.

While square waves and triangular waves will be explored in this textbook, the major type of time-varying source that will be used is a sinusoidal wave. It is important to recall the anatomy of a sine wave, shown below in figure 1.11.

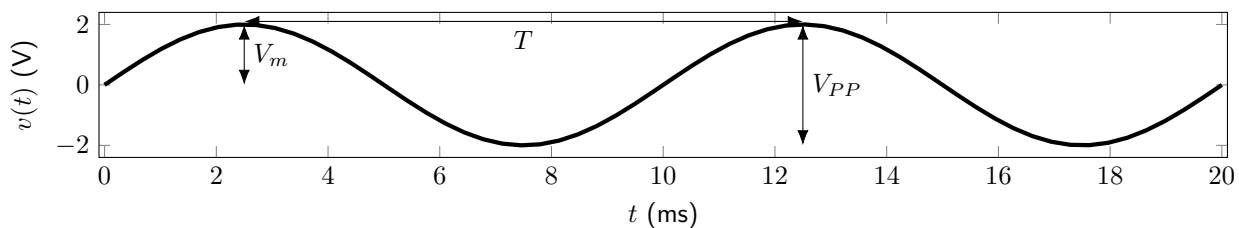


Figure 1.11: A timing diagram of a sinusoidal voltage source.

A cosine wave is expressed mathematically in equation 1.3.

$$v(t) = V_m \cos\left(\frac{2\pi t}{T} + \phi\right) = V_m \cos(2\pi ft + \phi) \quad (1.3)$$

While the equation is given in terms of voltage, it is also the same format that is used to describe current, power, or any other quantity that oscillates with respect to time. Each of the terms in the equation gives important information about the source.

- V_m is the **amplitude** of the source, which is defined as the maximum displacement from the equilibrium position.
- V_{PP} , while not part of the equation, describes the **peak-to-peak** value of the source. This is equal to twice the amplitude, or the difference between the maximum and minimum values.

- V_{DC} is not shown in either equation 1.3 or in figure 1.11, but is a constant added to a sine wave if the equilibrium of the wave is not equal to zero. This parameter is known as the **DC offset** of the wave.
- T is equal to the **period** of the wave, which describes how long it takes for one oscillation to occur.
- f is the wave **frequency**. This is the inverse of period ($f = 1/T$) and describes how many oscillations occur per unit of time.
- ϕ is the **phase** of the wave. It describes the angular offset of the start of the wave.

1.5.3 Dependent Sources

All of the sources described in the previous two subsections are independent sources. That is, the values of the sources are independent of any other property or characteristic of the circuit. This is the type of source that you would expect when using a power supply or a battery that has a given voltage rating. It doesn't matter what values exist elsewhere in the circuit; a AA battery will always supply a voltage of 1.5 V.

This subsection describes dependent sources. These sources deliver voltage and current proportionally to other circuit characteristics. It is important to note that dependent sources aren't generally describing something that you can buy and put into a circuit (such as a battery) but are used to model otherwise complicated circuit elements (such as operational amplifiers and transistors).

There are four types of dependent sources. The circuit symbols are shown in figure 1.12. Note that whereas independent sources are circular, dependent sources are diamond-shaped.

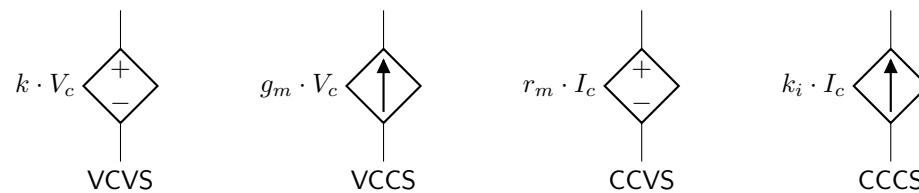


Figure 1.12: The four types of dependent sources: voltage-controlled voltage source, voltage-controlled current source, current-controlled voltage source, and current-controlled current source.

Each dependent source has a magnitude that's equal to a proportionality constant multiplied by a controlling value. Each of these values is described in table 1.1.

Source Type	Magnitude	Proportionality Constant	Controlling Value
Voltage-controlled voltage source (VCVS)	V_d (V)	k (unitless)	V_c (V)
Voltage-controlled current source (VCCS)	I_d (A)	g_m (A/V)	V_c (V)
Current-controlled voltage source (CCVS)	V_d (V)	r_m (V/A, Ω)	I_c (A)
Current-controlled current source (CCCS)	I_d (A)	k_i (unitless)	I_c (A)

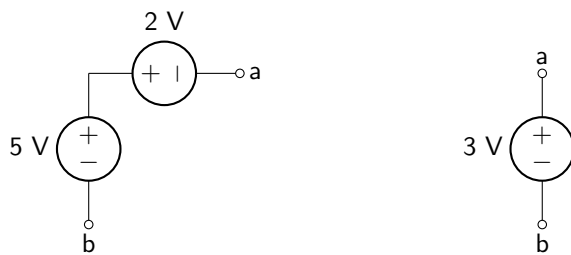
Table 1.1: Descriptions of the magnitude, proportionality constant, and controlling value for each type of dependent source.

1.5.4 Equivalent Sources

If a circuit contains multiple sources (whether they be DC or AC; dependent or independent), it may be possible to combine sources together into an equivalent source. This is the first tool that we can use to help simplify circuits.

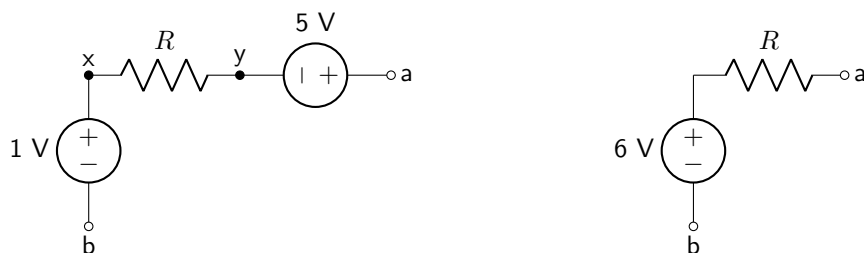
Voltage sources can be added together if they are in series with each other. If there are any other components between the sources, then the sources can still be added together as long as every circuit component is in series. If it is necessary to measure the voltage at a node in between sources, then the sources should not be combined.

Example: Combining voltage sources that are in series



The two voltage sources can be combined because they are in series with each other. The total voltage drop between nodes a and b are $-2\text{ V} + 5\text{ V} = 3\text{ V}$. The two voltage sources can therefore be combined into a single voltage source with the equivalent voltage of 3 V.

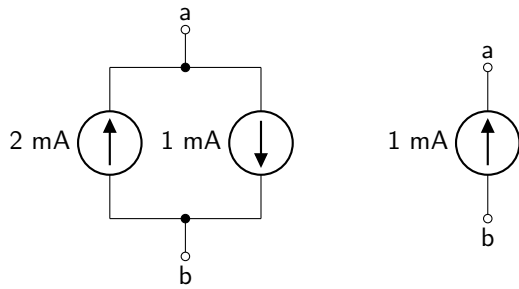
Example: Combining voltage sources that are in series with other components



Even though a resistor is placed between the two voltage sources, they can still be combined because the voltage sources are in series with each other. Note that if the voltage drop between node x and ground (or between node y and ground) needs to be measured, combining the voltage sources will lead to incorrect measurements.

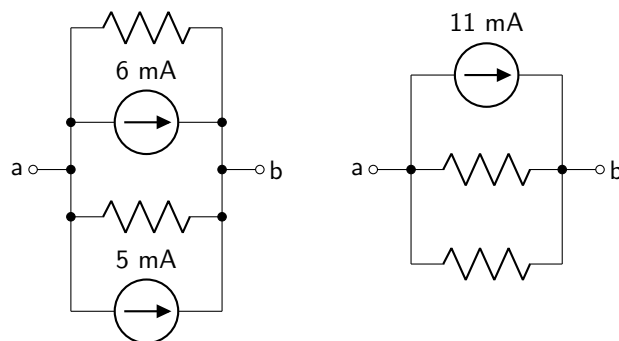
Current sources can be added together if they are in parallel with each other. If there are any other components between the sources, the sources can still be added together as long as every circuit component is in parallel.

Example: Combining current sources that are in parallel



The two current sources can be combined because they are in parallel with each other. The current flow from b to a is $2 \text{ mA} - 1 \text{ mA} = 1 \text{ mA}$. The two sources can therefore be combined into a single source with the equivalent current of 1 mA.

Example: Combining current sources that are in parallel with other components



The two current sources can be combined because they are in parallel with each other. The presence of the two resistors does not change anything about the configuration of the circuit. The total current flow from a to b is $6 \text{ mA} + 5 \text{ mA} = 11 \text{ mA}$.

1.6 Elementary Signals

Elementary signals are used to mathematically describe the types of signals used as sources, or signals that are commonly encountered in the analog circuit world. Each of the signals can be expressed mathematically.

1.6.1 Dirac Delta Function

The Dirac delta function is a useful approximation of a “spike.” It is used to model an impulse, and is commonly used in signal processing (including Fourier analysis). It is defined as a normal (Gaussian) distribution when the variance is equal to zero. The mathematical expression used to define the Dirac delta

function is given in equation 1.4, where A is the amplitude and τ is the time at which the function occurs.

$$f(t) = A\delta(t - \tau) \quad (1.4)$$

The Dirac delta function defined in equation 1.4 is depicted graphically in figure 1.13.

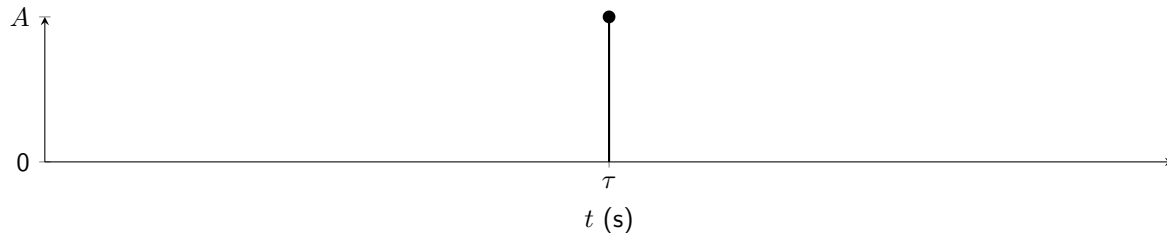
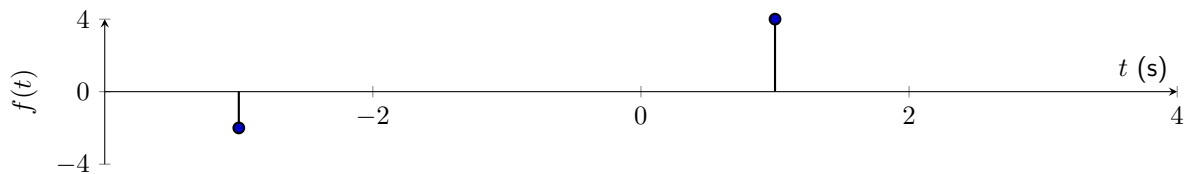


Figure 1.13: Graphical depiction of a Dirac delta function.

Example: Plotting a signal composed of Dirac delta functions

Plot $f(t) = 4\delta(t-1) - 2\delta(t+3)$. The first term has an amplitude of 4 and is located at $t = 1$ s. The second term has an amplitude of -2 and is located at $t = -3$ s.



The Dirac delta function has a *sifting property*. It is capable of being used to determine the value of a continuous function at a certain point. This property is defined in equation 1.5.

$$\int_{-\infty}^{\infty} f(t)\delta(t - a) dt = f(a) \quad (1.5)$$

1.6.2 Step Function

The unit step function is a function whose value is zero when the argument is negative, and one everywhere else. In general, the unit step function can be multiplied by a value to give it an amplitude that is greater than or less than one. The unit step function comes from integrating the Dirac delta function, as shown in equation 1.6.

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau \quad (1.6)$$

The step function is very useful for describing a source that is off for a long period of time before being turned on at some moment. Mathematically, the step function is defined in equation 1.7, where A is the

amplitude of the function and τ is the time at which the step starts.

$$f(t) = Au(t - \tau) \quad (1.7)$$

The step function defined in equation 1.7 is depicted graphically in figure 1.14.

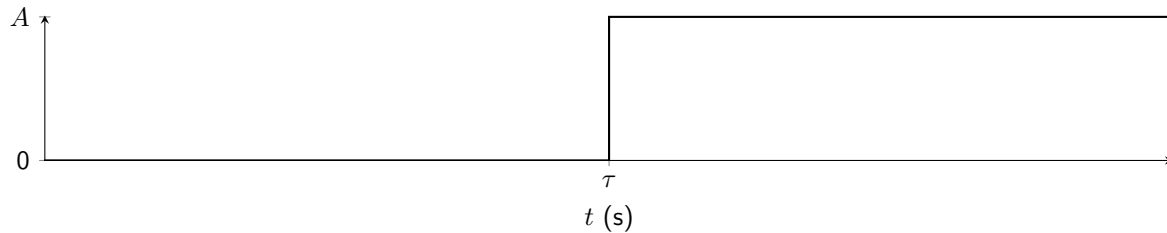
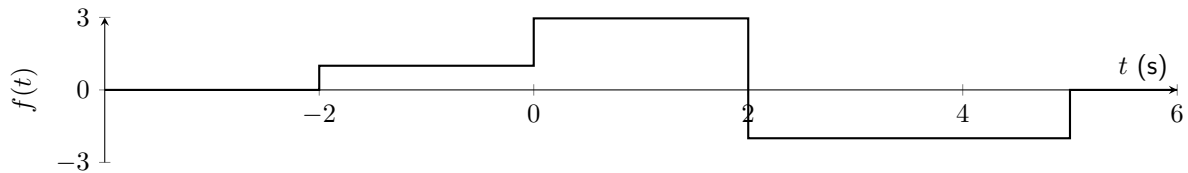


Figure 1.14: Graphical depiction of a step function.

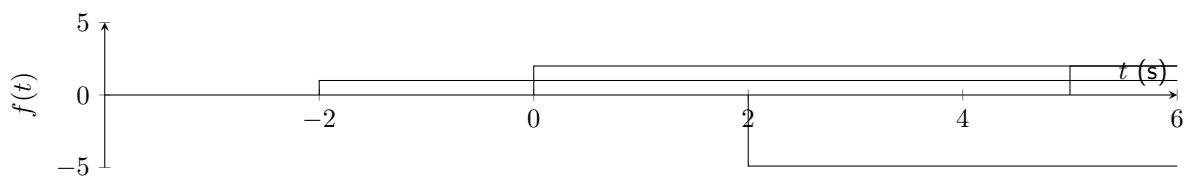
Example: Deriving the equation of a signal composed of step functions

Derive an equation corresponding to the graph below.



This signal is composed of four different step functions. The first occurs at $t = -2$ s and has an amplitude of 1. The second step function occurs at $t = 0$ s. It must have an amplitude of 2 in order to bring the total amplitude from 1 to 3. The third step function occurs at $t = 2$ s. Because the signal goes from a value of 3 to -2 , the step function must have an amplitude of negative 5. The last step function occurs at $t = 5$ s and has an amplitude of positive 2 to bring the signal value back to zero.

Each of these individual step functions is depicted below. The sum of each of the step functions will lead to the graph above.



$$f(t) = u(t+2) + 2u(t) - 5u(t-2) + 2u(t-5)$$

1.6.3 Ramp Function

The ramp function is a signal whose value is zero when the argument is negative, and then has a constant linear upward slope when the argument is positive. The ramp function comes from integrating the step

function, as shown in equation 1.8.

$$r(t) = \int_{-\infty}^t u(\tau) d\tau \quad (1.8)$$

The equation used to express the ramp function is given in equation 1.9, where A is the slope of the ramp, and τ is the time that the ramp function initiates.

$$r(t) = A(t - \tau)u(t - \tau) \quad (1.9)$$

The graphical depiction of a ramp function is shown in figure 1.15.

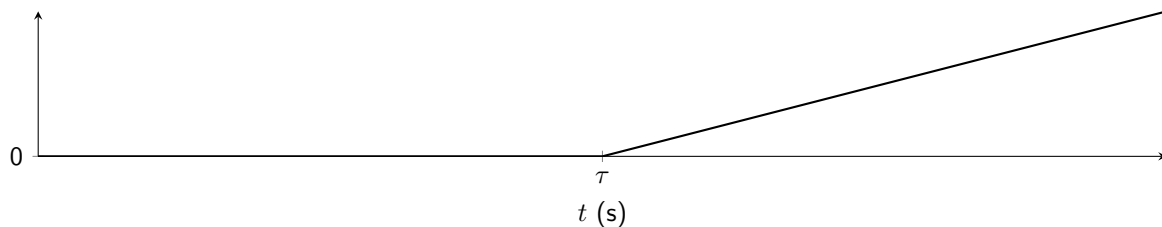
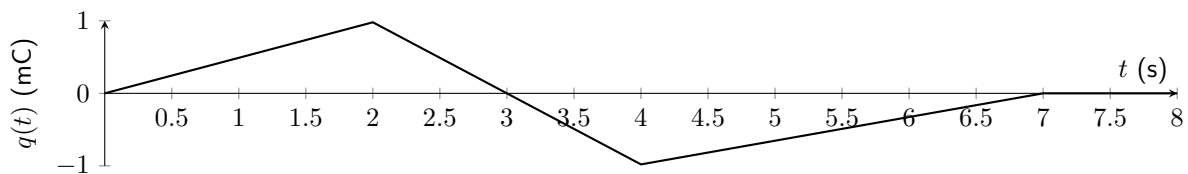


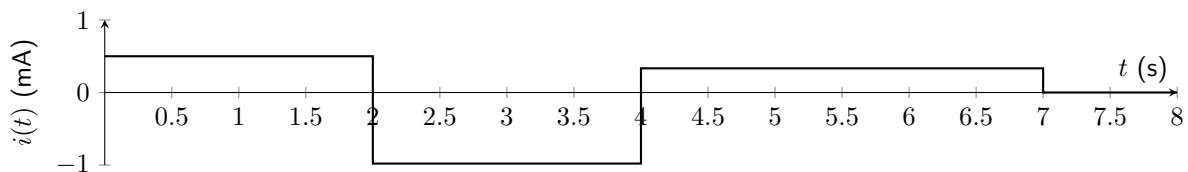
Figure 1.15: Graphical depiction of a ramp function.

Example: Calculating current from ramp function charge

The charge entering a circuit element is shown in the graph below. Use it to calculate the current flow.



Current is equal to the time derivative of charge. The time derivative is plotted below.



$$i(t) = 0.5 \text{ mA } u(t) - 1.5 \text{ mA } u(t-2) + 1.3 \text{ mA } u(t-4) - 0.3 \text{ mA } u(t-7)$$

1.6.4 Rectangular Pulse

A rectangular pulse describes a signal that is initially zero, takes on a particular value for a finite interval of time, and then returns instantaneously to zero. It is different from the step function in that the step function never returns to zero again. A rectangular pulse is especially relevant in electrical engineering as it can be used to describe periodic square waves that are commonly used as clock sources. The mathematical

equation for a rectangular pulse is given in equation 1.10, where A defines the amplitude of the pulse, τ defines the width of the pulse, and λ defines the center of the function.

$$f(t) = A \operatorname{rect}\left(\frac{t - \lambda}{\tau}\right) \quad (1.10)$$

A rectangular pulse is depicted graphically in figure 1.16.

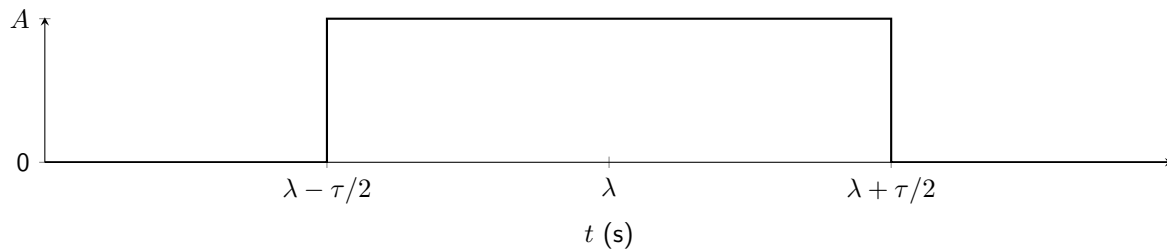
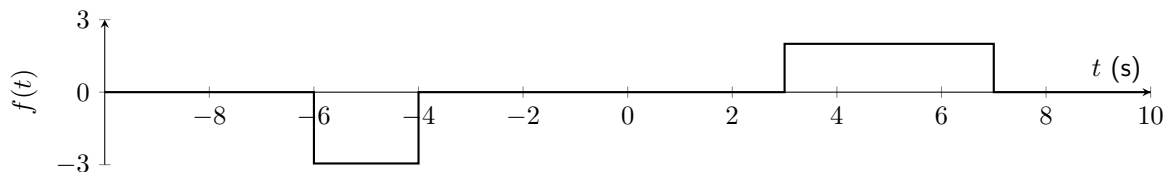


Figure 1.16: Graphical depiction of a rectangular pulse function.

Example: Plotting a signal composed of rectangular pulses

Plot $f(t) = -3\operatorname{rect}((t+5)/2) + 2\operatorname{rect}((t-5)/4)$. The first term has an amplitude of -3 , a width of 2 s, and is centered at $t = -5$ s. The second term has an amplitude of 2 , a width of 4 s, and is centered at $t = 5$ s.



1.6.5 Triangular Pulse

The triangular pulse is defined as a function that is zero, linearly ramps up to a particular value, and then linearly ramps down to zero again. As the rectangular pulse is to the step function; the triangular pulse is to the ramp function. The equation for a triangular pulse is given in equation 1.11, where A is the amplitude of the pulse, τ is the half-width of the pulse, and λ is the center of the pulse. Note that the definitions for the width of a rectangular and triangular pulse are different!

$$f(t) = A \operatorname{tri}\left(\frac{t - \lambda}{\tau}\right) \quad (1.11)$$

A triangular pulse is shown graphically in figure 1.17.

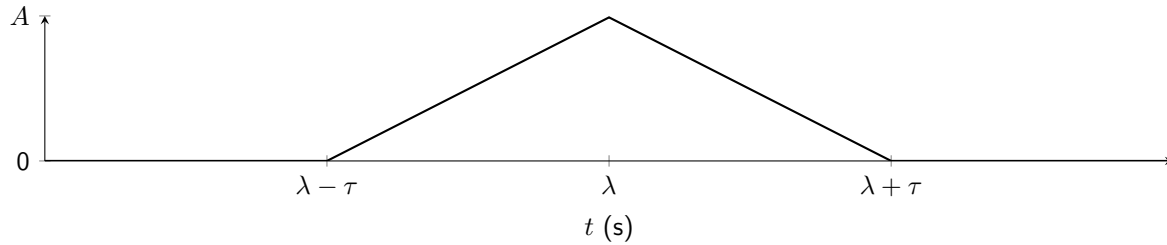
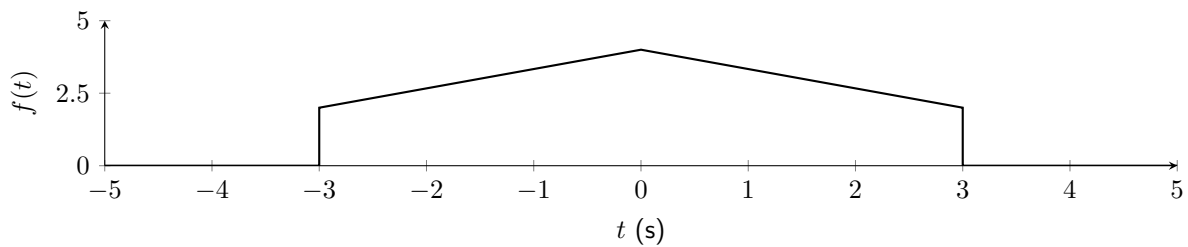


Figure 1.17: Graphical depiction of a triangular pulse function.

Example: Finding the equation of a waveform

Find the equation corresponding to $f(t)$, shown below.



$f(t)$ is composed of a rectangular pulse and a triangular pulse. The rectangular pulse has an amplitude of 2, is centered at $t = 0$ s, and has a width of 6 s. The triangular pulse has an amplitude of 3, is centered at $t = 0$ s, and has a half-width of 3 s.

$$f(t) = 2\text{rect}(t/6) + 3\text{tri}(t/3)$$

1.6.6 Exponential Decay

The exponential function is ubiquitous in engineering. Specifically, a decaying exponential is found as the solution to many engineering problems. (A rising exponential function is not stable as it would eventually require an infinite amount of energy to sustain.) An exponential can be defined mathematically in equation 1.12, where A is the initial amplitude of the exponential, τ is the time at which the exponential begins its decay, and α is how quickly the exponential decays to zero.

$$f(t) = Ae^{-\alpha(t-\tau)}u(t-\tau) \quad (1.12)$$

A decaying exponential function is depicted graphically in figure 1.18.

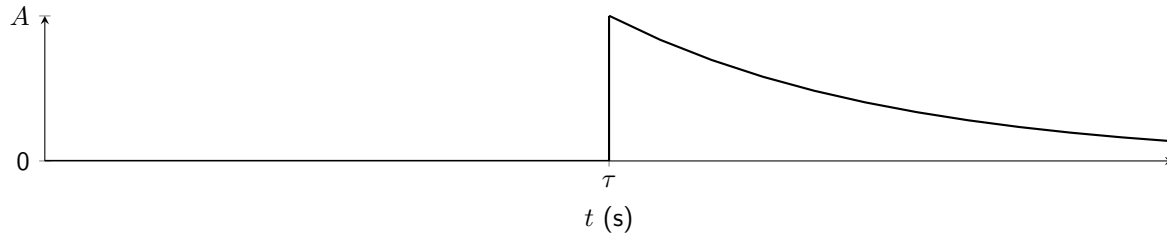


Figure 1.18: Graphical depiction of exponential decay.

1.6.7 Damped Sinusoid

A damped sinusoid describes the behavior of almost all harmonic motion (such as a mass on a spring, a door swinging open and closed on a hinge, or a second-order circuit). A damped sinusoid is defined as a sinusoidal signal multiplied by a decaying exponential, as shown in equation 1.13

$$f(t) = Ae^{-\alpha(t-\tau)} \sin(\omega t)u(t - \tau) \quad (1.13)$$

In equation 1.13, all of the terms are the same as those given for the exponential decay. ω defines the angular frequency of the sinusoidal wave. A damped sinusoid is shown graphically in figure 1.19.

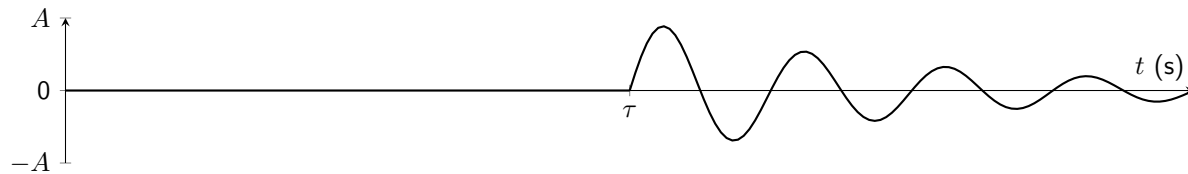


Figure 1.19: Graphical depiction of a damped sinusoid.

Example Problems

Power

1. How much power is supplied by the unknown element shown in figure 1.20?

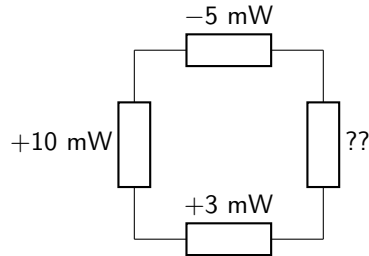


Figure 1.20: Circuit schematic for power question 1.

2. A voltage source supplies 12 V and has a power consumption of 5 W. How much current is the voltage source supplying to the circuit?
3. Which of the circuit elements shown in figure 1.21 (a) absorb power and (b) deliver power? Current flows clockwise through the circuit.

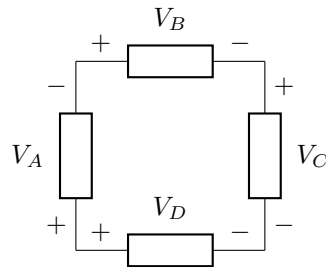


Figure 1.21: Circuit schematic for power question 3.

4. Calculate the power consumed by a circuit when $v(t) = 5 \cos(2\pi 50t)$ V and $i(t) = 0.1 \cos(2\pi 50t)$ A.
5. Calculate the power consumed by a circuit when $v(t) = 3t$ V and $i(t) = 40\delta(t - 5)$ mA.

Sinusoidal Waves

6. What are V_m , V_{DC} , f , and ϕ of the function $v(t) = 3.8 \cos(2\pi 20t + 25\pi/180) + 3$ V?
7. What are V_m , V_{PP} , and T of the function $v(t) = 2.19 \cos(2\pi 40t - 20\pi/180) + 6$ V?

8. What are V_m , V_{DC} , and f of the function shown in figure 1.22?

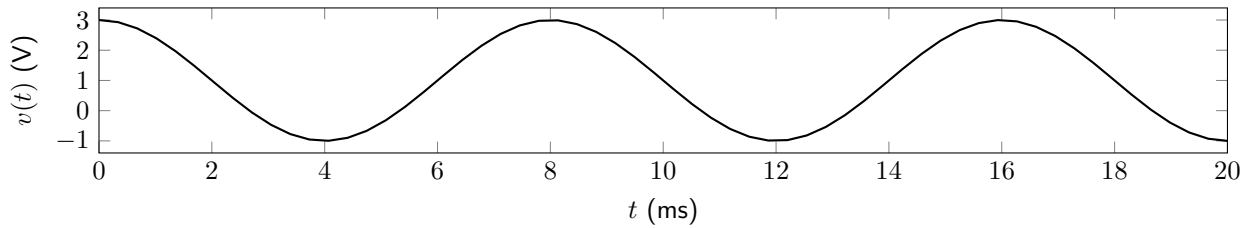


Figure 1.22: Waveform for sinusoidal waves question 8.

9. Sketch the function $i(t) = 400 \cos(2\pi 100t + 45\pi/180) + 200$ mA.
10. What minimum DC offset should be added to $i(t) = 30 \cos(2\pi 200t - 50\pi/180)$ mA to prevent any negative values from occurring on the output?

Sources

11. What kind of dependent source is shown in figure 1.23? How do you know?

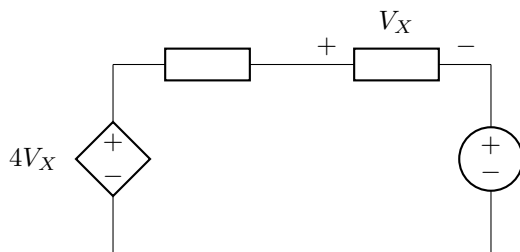


Figure 1.23: Circuit schematic for sources question 11.

12. What kind of dependent source is shown in figure 1.24? How do you know?

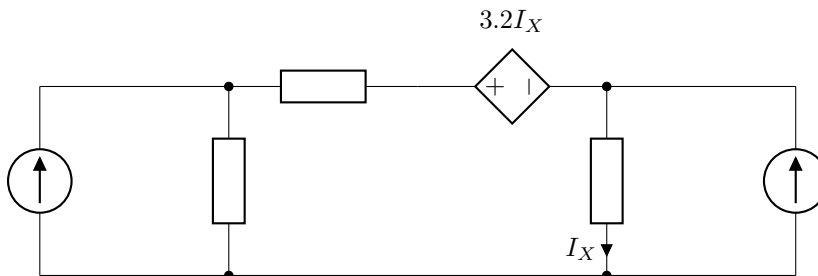


Figure 1.24: Circuit schematic for sources question 12.

13. Calculate the current supplied by the dependent source as shown in figure 1.25.

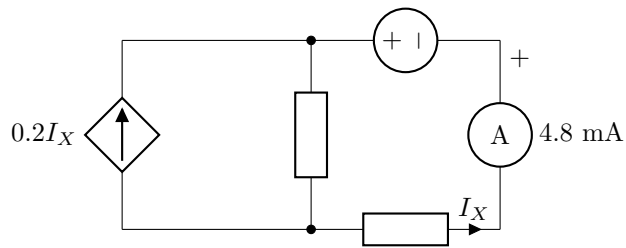


Figure 1.25: Circuit schematic for sources question 13.

14. Reduce figure 1.26 to a circuit that contains only one independent source.

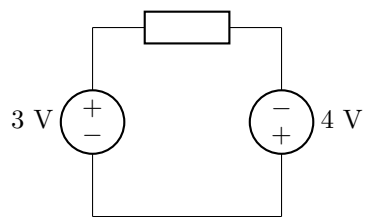


Figure 1.26: Circuit schematic for sources question 14.

15. Reduce the circuit given in figure 1.27 as much as possible.

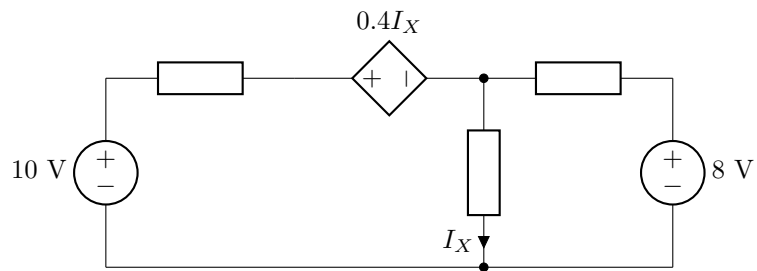


Figure 1.27: Circuit schematic for sources question 15.

Elementary Signals

16. Find an equation for the signal shown in figure 1.28.

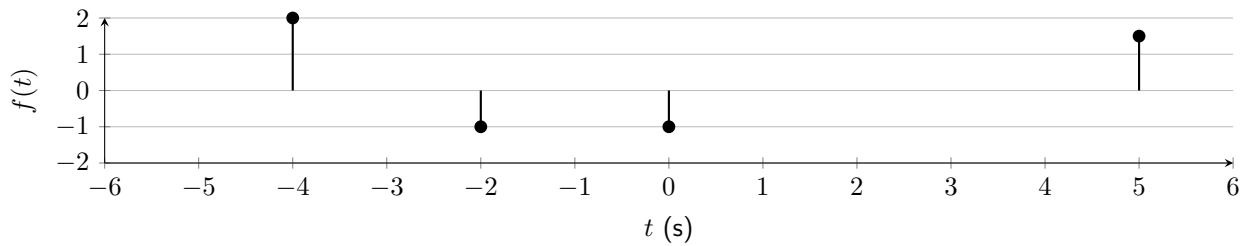


Figure 1.28: Signal for elementary signals question 16.

17. Find an equation for the signal shown in figure 1.29.

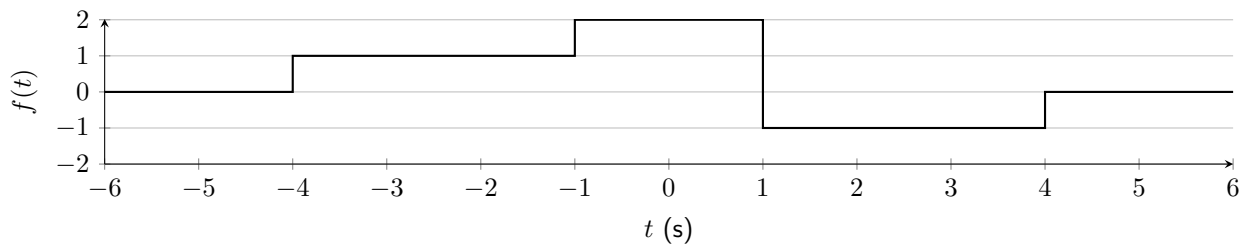


Figure 1.29: Signal for elementary signals question 17.

18. Sketch the function $f(t) = 3 \operatorname{tri}(t + 2) - 2 \operatorname{tri}\left(\frac{t-5}{5}\right)$.

19. Sketch the function $f(t) = 3u(t + 5) + \frac{t}{2}u(t)$.

20. If the charge in a circuit is measured as $q(t) = 2 \operatorname{tri}\left(\frac{t+6}{2}\right) + 0.5tu(t) - 1.5(t-4)u(t-4) + 1.5(t-8)u(t-8) - 0.5(t-12)u(t-12)$ C, sketch the current.

2 Circuit Laws

Circuit laws enable us to fill up our “toolbox” with tools that enable us to simplify and analyze circuits. First, we must understand the terminology used in circuits.

2.1 Definitions

A **circuit** is an interconnection of elements. It can be simple (consisting of very few parts), very complicated, or anywhere in between. Circuit elements consist of sources, passive elements, and active elements. Sources were described in section 1.5 in the previous chapter. They can be dependent or independent and can source either voltage or current. Passive elements consist of components such as resistors, capacitors, inductors, and diodes. These circuit elements do not require a connection to a power supply in order to function. A resistor inherently has resistance, which does not need to be activated with a power supply. Active elements, on the other hand, require power to operate. Transistors and operational amplifiers are examples of active circuit elements.

A **node** is a point in a circuit where two or more elements are connected. It is important to recognize and understand nodes, as they are used in many of the circuit analysis tools explained in this chapter. A diagram depicting three types of circuit nodes (depicted with thick black dots) is shown in figure 2.1. The rectangles in each diagram represents a “generic” circuit element (which could be a resistor, a capacitor, a source, or something else).

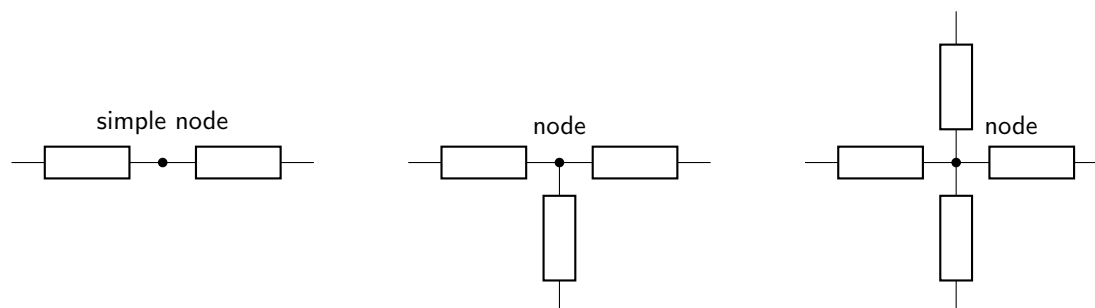


Figure 2.1: Three examples of circuit nodes.

It is important to note that a node is not always depicted with a thick black dot on a circuit diagram. A node exists at a point of common potential. The circuit shown in figure 2.2 has three nodes labeled A, B, and C. Node A is a point of common potential between the power supply and element 1. Node B is a point of common potential connecting elements 1, 2, and 3. This node looks like the nodes shown in figure 2.1, as it is depicted with a thick dot (indicating an electrical connection). Finally, node C is a point of common potential connecting the low potential end of the power supply, one end of element 2, and one end of element 3.

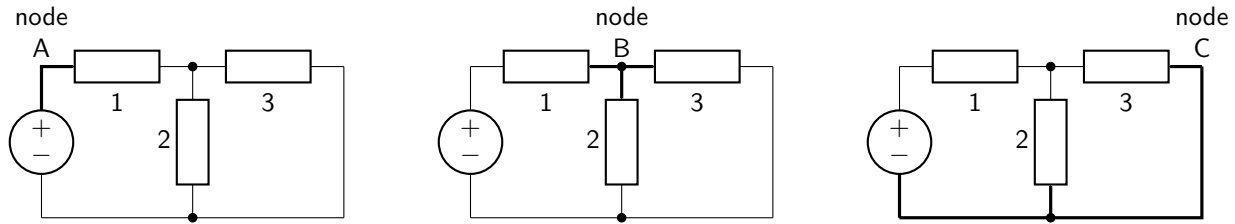


Figure 2.2: Each of the three nodes in this circuit is shown with thick lines. Node A (left) connects the power supply and element 1. Node B (middle) connects elements 1, 2, and 3. Node C (right) connects the power supply to elements 2 and 3.

In figure 2.2, we can say that elements 2 and 3 share two common nodes (nodes B and C). Later in this chapter we will learn that devices that are connected with two common nodes are said to be in parallel with each other.

A **branch** in a circuit represents a portion of a circuit that exists in a path between two nodes. A **loop** is a closed path that starts at a node and returns to the same node. Finally, a **mesh** is a loop that has no other loops inside of it. Nodes, loops and meshes are important concepts, as they are used in circuit analysis tools explained later in this chapter. The circuit diagram shown in figure 2.3 has six branches: one for each circuit element (sinusoidal source, DC voltage source, DC current source, resistor, capacitor, and inductor).

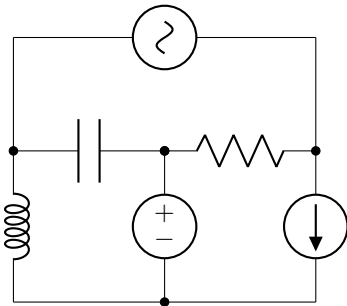


Figure 2.3: A circuit diagram consisting of four nodes, six branches, seven loops, and three meshes.

The three meshes of the circuit in figure 2.3 consist of loops that do not have any other loops inside of them. These are shown in thick outlines in figure 2.4.

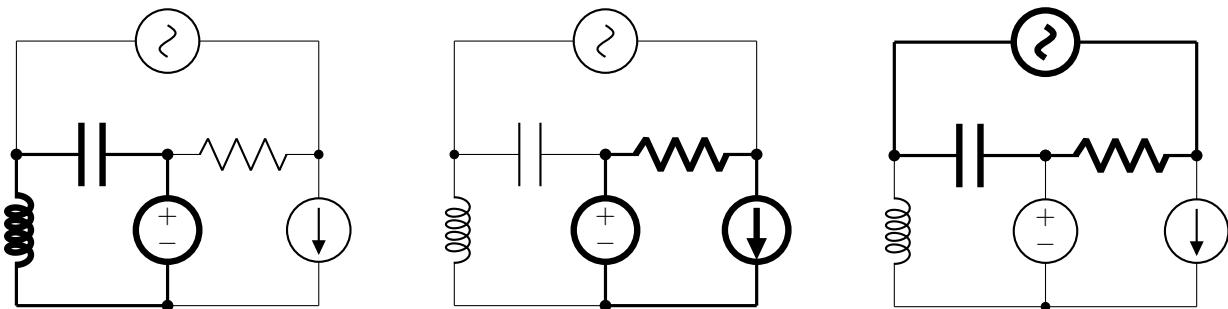


Figure 2.4: The three meshes (each shown in thick lines) from the circuit in figure 2.3.

The seven loops in the circuit include the three meshes as well as four additional loops, each of which contain other loops inside of them. These four additional loops are shown in figure 2.5.

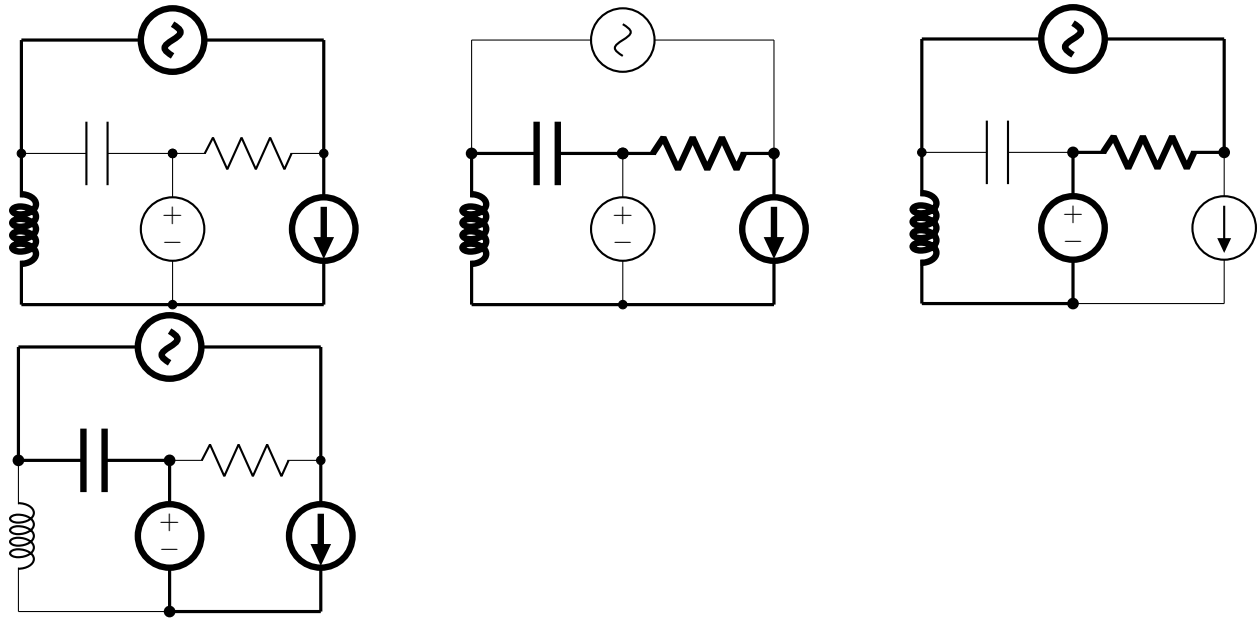


Figure 2.5: The four additional circuit loops (each shown in thick lines) from the circuit in figure 2.3.

2.2 Resistor

A resistor is a circuit element that regulates the flow of current. This is the first passive element that will be studied in detail in this textbook. Resistance, which is a property of resistors, defines how much the resistor opposes the flow of current. Resistance has a symbol of R and units of Ω (ohms), and is given by equation 2.1, where ρ is a property inherent to a material (known as resistivity), l is the length of the resistor, and A is the cross-sectional area of the resistor.

$$R = \frac{\rho l}{A} \quad (2.1)$$

The symbol used for a resistor in circuit diagrams is shown in figure 2.6.



Figure 2.6: The circuit symbol for a resistor.

Resistors have very many important applications in circuits (tuning resonant frequencies, creating particular amplifier gains, or to create particular voltage or current values, among other things), and are therefore an important circuit element to understand.

Physically, resistors come in many shapes, sizes, and forms. There are fixed resistors (resistors whose resistance value is not meant to change) and variable resistors (which will be discussed in more detail in

section 2.2.2 below). Most typical hobbyist resistors are made from carbon film deposited on a piece of ceramic, with the ends of the carbon film connected to metal wires (known as leads). A helix shape is carved into the carbon film; it is the pitch of the helix and thickness of the carbon film that determines the resistance of the device. Carbon film resistors typically have tolerances between 2%–5%.

2.2.1 Resistor Color Codes

Carbon film resistors are coated and painted with colored stripes to distinguish each resistor's value in ohms. Most of these resistors contain four stripes (although five- and six-stripe versions do exist, they will not be discussed in this book). These stripes are depicted in figure 2.7. The first and second bands dictate the value of the resistance. The third band acts as a multiplier, and the tolerance band denotes how accurate the value of the resistor is. Each of the colors depicts a particular value, outlined in table 2.1.

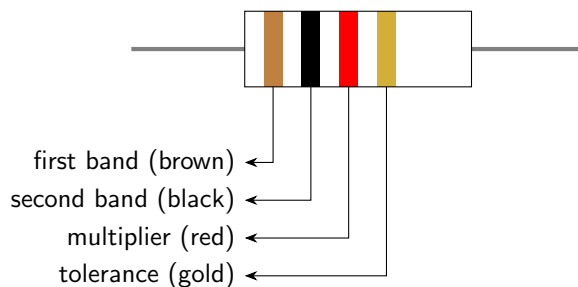


Figure 2.7: Depiction of each of the colored bands on a four-band carbon film resistor. This particular resistor has brown-black-red-gold stripes, and therefore has a value of $1000 \Omega \pm 5\%$.

Color	Value (1st and 2nd bands)	Multiplier	Tolerance
Black	0	$10^0 \Omega$	–
Brown	1	$10^1 \Omega$	$\pm 1\%$
Red	2	$10^2 \Omega$	$\pm 2\%$
Orange	3	$10^3 \Omega$	$\pm 0.05\%$
Yellow	4	$10^4 \Omega$	$\pm 0.02\%$
Green	5	$10^5 \Omega$	$\pm 0.5\%$
Blue	6	$10^6 \Omega$	$\pm 0.25\%$
Violet	7	$10^7 \Omega$	$\pm 0.1\%$
Gray	8	–	$\pm 0.01\%$
White	9	–	–
Gold	–	$10^{-1} \Omega$	$\pm 5\%$
Silver	–	$10^{-2} \Omega$	$\pm 10\%$
no band	–	–	$\pm 20\%$

Table 2.1: Descriptions of each of the colors used to denote values on a resistor.

To determine the value of a resistor, it is important to first establish which band corresponds to the first, and which corresponds to the tolerance band. Usually, the first band is painted closer to the lead than the tolerance band is. Otherwise, it can be helpful to note that the large majority of inexpensive hobbyist carbon-film resistors have a gold tolerance band, which also can help orient the resistor (but this trick will

not be valid with any other tolerance value!). The first and second bands give the value of the resistor. Red orange is 23. Brown green is 15. Then, the multiplier is used to determine the weight of the value. Red orange blue is $23 \times 10^6 \Omega$, or 23 M Ω . Brown green red is $15 \times 10^2 \Omega$, or 1.5 k Ω . The tolerance band depicts the range of percentages over which the actual value of the fabricated resistor is valid.

2.2.2 Variable Resistors and Potentiometers

A variable resistor is a resistor whose value is not fixed but that can be altered by the user. Variable resistors used to be called *rheostats*, but that term is falling out of favor. The circuit symbol used for variable resistors is shown in figure 2.8. Note that a variable resistor is a two-terminal device.



Figure 2.8: Circuit symbol used for a variable resistor.

A potentiometer is a three-terminal device that can be used as either a voltage divider or a variable resistor. The circuit symbol for a potentiometer is shown in figure 2.9.



Figure 2.9: Circuit symbol used for a potentiometer.

Two fixed leads are connected to a resistive element (commonly made out of either carbon or wound wire) in a semi-circular configuration. A moveable lead called the *wiper* can then be positioned at any point along the resistive element. This is depicted in figure 2.10. While the total resistance between the fixed leads is constant, the resistance between the wiper and either of the two fixed leads changes as the wiper's position changes.

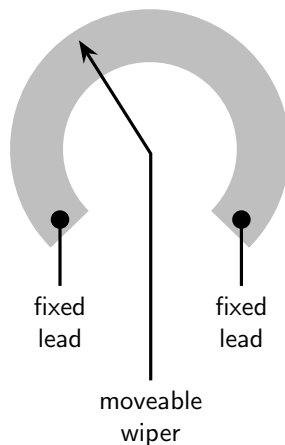


Figure 2.10: A schematic of a potentiometer. The gray semicircular area corresponds to the resistive element. Two leads (one at each end of the resistive element) are fixed. The third lead is a moveable wiper.

A potentiometer can be used as a voltage divider. Voltage dividers (which will be explained in greater detail in section 2.5) take a fixed voltage and proportionally divide it up into two or more values. If a fixed voltage is applied to the two fixed leads of a potentiometer, then the voltage drop between either lead and the wiper will be modified depending on the position of the wiper. This is depicted schematically in figure 2.11. Note that while V_1 and V_2 can change depending on the position of the wiper, $V_1 + V_2 = V_S$ will always be satisfied.

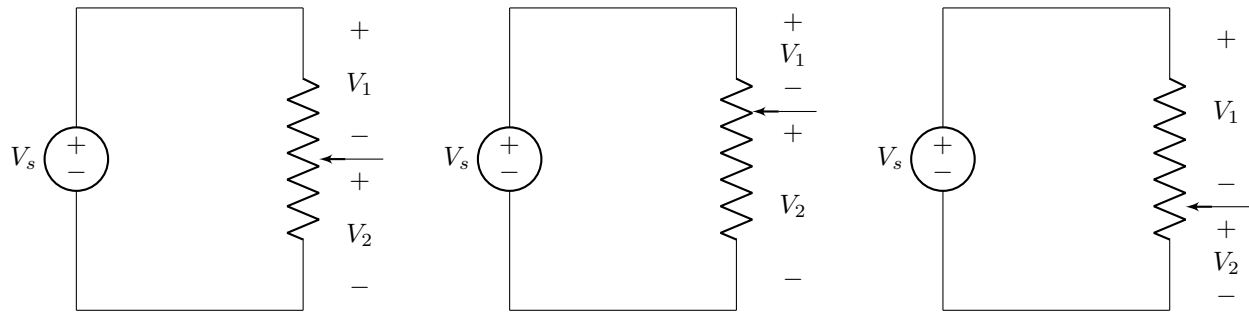


Figure 2.11: A potentiometer can be used as a voltage divider. The relationship $V_1 + V_2 = V_S$ is always preserved, regardless of the individual values of V_1 and V_2 .

A potentiometer can also be used as a variable resistor. In this case, one of the fixed leads will be ignored, and the resistance between the other fixed lead and the wiper will be used. Because the wiper itself has a small amount of resistance, it should be shorted to the unused fixed lead. This is depicted schematically in figure 2.12.

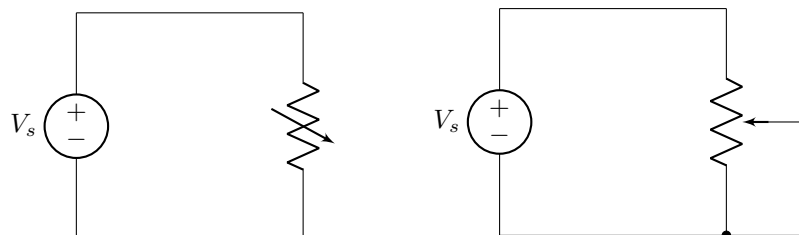


Figure 2.12: A potentiometer can be used as a variable resistor by configuring it as shown in the right-hand diagram. Both of these schematics are functionally equivalent.

2.2.3 Measuring Resistance

Resistance can be measured in a circuit using a device called an ohmmeter. The circuit schematic for an ohmmeter is shown in figure 2.13. An ohmmeter contains two leads. When the leads are placed on either end of the elements of a circuit, a constant current is passed through the circuit elements and the resulting voltage is measured with an analog to digital converter. Using Ohm's law, the resistance of the circuit elements can be calculated and displayed. Because resistors don't have any polarity or direction to them, the ordering of the leads is unimportant (the common lead can be placed on either end of the circuit elements without impacting the result). However, because an ohmmeter operates by sending a current through the circuit

elements and measuring the resulting voltage, it is important to know that resistance cannot be measured in a circuit unless all other power sources have been disconnected.

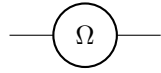


Figure 2.13: Circuit symbol used for an ohmmeter.

Because an ohmmeter sends a constant current through a circuit and measures the resulting voltage, an ohmmeter must be connected in parallel across the components to be measured, as shown in figure 1.5.

2.2.4 Resistors in Series

The first tool to analyze and simplify resistors is to understand how to combine resistors in series. That is: all of the resistors are in the same path in a circuit and share the same current flow. A circuit diagram containing five resistors in series is shown in figure 2.14.



Figure 2.14: Five resistors in series in a circuit.

Without having formalized the concepts of Kirchhoff's laws yet (which will occur in section 2.7), we can use equation 2.1 to form a conceptual understanding of what happens when resistors are connected together in series. Of the parameters given in equation 2.1, the only one that effectively changes when resistors are connected in series is the length (l). The resistivity, being a property inherent to the material used to create the resistor does not change. The cross-sectional area also remains unchanged throughout. This is depicted in figure 2.15.

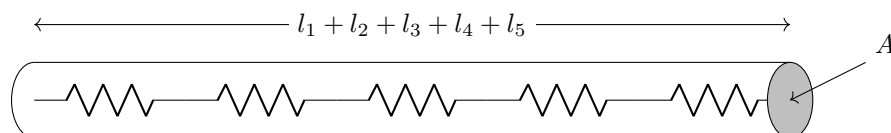


Figure 2.15: When resistors are connected in series, the parameter of equation 2.1 that changes is the length.

Because the total length of the circuit is equal to the sum of the individual lengths of each resistor, we can express the total resistance (known as the **equivalent resistance**) of the circuit likewise, shown in equation 2.2 and derived below.

$$R_{EQ} = R_1 + R_2 + R_3 + \dots + R_n \quad (2.2)$$

DERIVATION

$$\begin{aligned}
 R_{EQ} &= \frac{\rho(l_1 + l_2 + l_3 + \dots + l_n)}{A} \\
 &= \frac{\rho l_1}{A} + \frac{\rho l_2}{A} + \frac{\rho l_3}{A} + \dots + \frac{\rho l_n}{A} \\
 &= R_1 + R_2 + R_3 + \dots + R_n
 \end{aligned}$$

Qualitatively, the resistance in a series circuit increases. It is therefore impossible for the equivalent resistance of a series circuit to be less than the value of any individual resistor. (This last property acts as a useful “smell test” when working out complicated circuits problems.)

2.2.5 Resistors in Parallel

The second tool that can be used to simplify complicated resistive circuits is equivalent resistance for resistors in parallel. In this case, each resistor shares a common node on either end of the circuit element, but each contains its own unique branch through which current can flow. A circuit diagram containing five resistors in parallel is shown in figure 2.16.

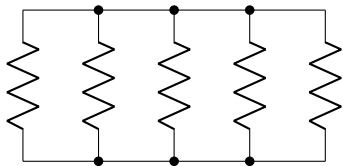


Figure 2.16: Five resistors in parallel in a circuit.

In this case, the effective parameter that changes in equation 2.1 is the cross-sectional area (A), shown in figure 2.17. The length and resistivity remain unchanged.

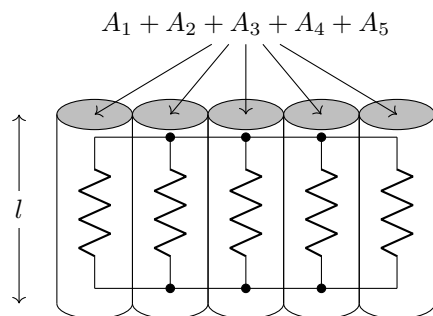


Figure 2.17: When resistors are connected in parallel, the parameter of equation 2.1 that changes is the area.

It is now possible to express the equivalent resistance of a circuit with parallel resistors using the equation

shown in equation 2.3 and derived below.

$$\frac{1}{R_{EQ}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots + \frac{1}{R_n} \quad (2.3)$$

DERIVATION

$$\begin{aligned} R_{EQ} &= \frac{\rho l}{A_1 + A_2 + A_3 + \dots + A_n} \\ &= \frac{1}{\frac{A_1}{\rho l} + \frac{A_2}{\rho l} + \frac{A_3}{\rho l} + \dots + \frac{A_n}{\rho l}} \\ \frac{1}{R_{EQ}} &= \frac{A_1}{\rho l} + \frac{A_2}{\rho l} + \frac{A_3}{\rho l} + \dots + \frac{A_n}{\rho l} \\ &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots + \frac{1}{R_n} \end{aligned}$$

Qualitatively, the resistance in a parallel circuit decreases. The equivalent resistance cannot be greater than the value of any of the individual resistors.

Frequently, it is necessary to combine two resistors in parallel. In that case, we can solve equation 2.3 to find a handy equation to use without having to calculate the reciprocal of the sum of reciprocals. (Note: this equation will **only** work for two resistors in parallel.) This solution is shown in equation 2.4 and is derived below.

$$R_{EQ} = \frac{R_1 R_2}{R_1 + R_2} \quad (2.4)$$

DERIVATION

$$\begin{aligned} \frac{1}{R_{EQ}} &= \frac{1}{R_1} + \frac{1}{R_2} \\ &= \frac{R_2}{R_1 R_2} + \frac{R_1}{R_1 R_2} \\ &= \frac{R_1 + R_2}{R_1 R_2} \\ R_{EQ} &= \frac{R_1 R_2}{R_1 + R_2} \end{aligned}$$

2.2.6 Equivalent Resistance

It is important to point out that most circuits are not 100% series nor 100% parallel. Many circuits have segments that contain elements in parallel, and segments that contain elements in series. In our discussion of delta-wye transforms in the next section in this book, we will see that there are some resistor combinations that are neither parallel nor series, but require a transformation before they can be made into parallel or

series combinations.

The equivalent resistance in a circuit refers to the amount of resistance “seen” between two particular nodes. (While we know that there are no eyeballs looking between nodes – at least, there are none depicted on the circuit diagram – this phrase is commonly used to describe the calculation of equivalent resistance between two nodes.) The equivalent resistance “seen” between two nodes indicates the resistance that would be measured between the two nodes.

Equivalent resistance can be calculated by combining resistors using the parallel and series combinations explained previously. Usually this is an iterative process. Start by combining any resistors that are *purely* in parallel with each other, or that are *purely* in series with each other. Then, reassess the circuit and see if any more opportunities for combinations arise. Continue until the circuit contains as few elements as possible.

The symbology that will be used in equivalent resistance equations in this textbook is a $+$ sign for series combination (as the resistances are indeed just being added together) and a $//$ symbol for parallel combinations. The order of operations is to deal with everything inside of parenthesis first; after that calculate parallel combinations; finally, calculate series combinations.

Consider the circuit shown in figure 2.18. Note that the circuit is not purely series nor is it purely parallel. In addition, we cannot start combining together circuit elements until we know which nodes to use to calculate the equivalent resistance. It is possible to calculate the equivalent resistance between any two nodes in the circuit. Not all of the circuit nodes are shown; just a few have been selected to demonstrate different equivalent resistance calculations.

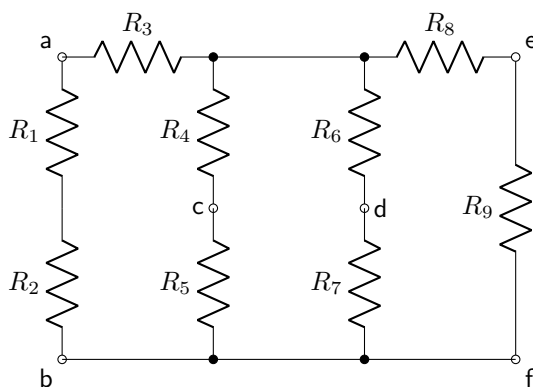


Figure 2.18: A circuit composed of nine resistors that is neither purely series nor purely parallel.

While it might be tempting to immediately combine resistors R_4 and R_5 in series, resistors R_6 and R_7 in series, and then those two combinations in parallel with each other, it is possible to calculate an equivalent resistance between nodes c and d ; in that case, those resistors cannot be combined! Therefore, it is important to pay attention to the nodes to be used to calculate the equivalent resistance before combining resistors together. We will use this circuit to calculate equivalent resistance between three different pairs of nodes to demonstrate the process. In all cases, it is wise to start combining resistors as far away from the reference nodes as possible.

Schematically, it's possible to depict equivalent resistance measured between two nodes by clearly indicating them on a circuit diagram. This is shown for the calculation of equivalent resistance between nodes a and b in figure 2.19.

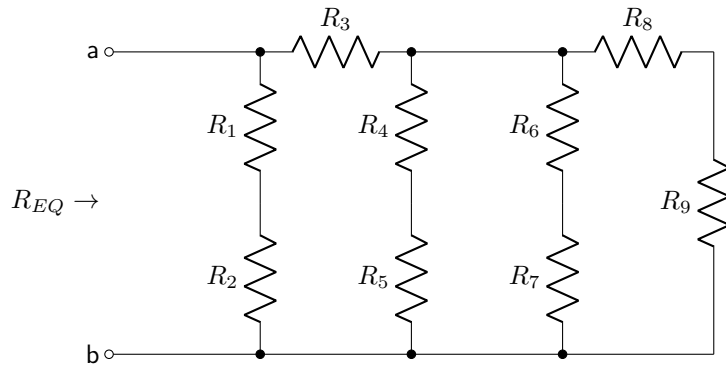


Figure 2.19: Schematic depiction of the equivalent resistance between nodes a and b in the circuit given in figure 2.18.

In this case, the resistors on the right-hand side of the schematic are as far from the reference nodes as possible. First, combine resistors R_8 and R_9 in series, as they share a common path in the circuit. Resistors R_4 and R_5 can be combined in series, as can resistors R_6 and R_7 . Those equivalent resistances can now be combined in parallel. That combination can be added in series with R_3 . Finally, that equivalent resistance can be combined in parallel with the series combination of R_1 and R_2 . This equivalent resistance is shown mathematically in equation 2.5. If each of the resistors were equal to 1 k Ω , then the equivalent resistance would be 10/11 k Ω .

$$R_{EQ,ab} = ((R_8 + R_9) // (R_6 + R_7) // (R_4 + R_5) + R_3) // (R_1 + R_2) \quad (2.5)$$

Next, let's consider the case of calculating the equivalent resistance between nodes e and f. This is depicted schematically in figure 2.20.

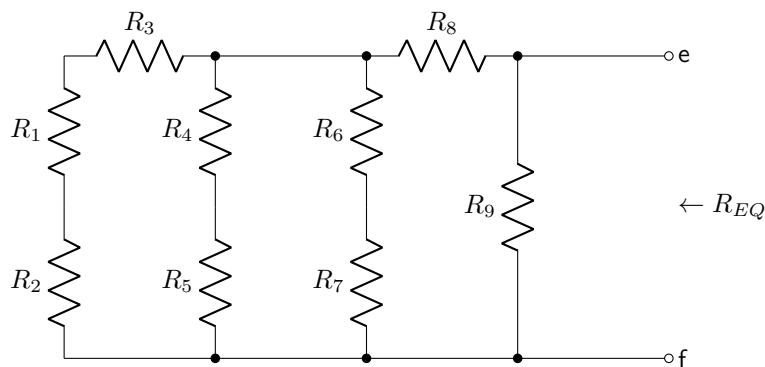


Figure 2.20: Schematic depiction of the equivalent resistance between nodes e and f in the circuit given in figure 2.18.

In this case, the resistors on the left-hand side of the schematic are as far from the reference nodes as possible. Resistors R_1 , R_2 , and R_3 can be combined in series. That equivalent resistance can then be combined in parallel with the series combinations of R_4 and R_5 as well as R_6 and R_7 . That equivalent resistance is combined in series with R_8 . That equivalent resistance is finally combined in parallel with R_9 . This is shown mathematically in equation 2.6. If each of the resistors were equal to 1 k Ω , the equivalent resistance would be 7/11 k Ω . Equivalent resistance is not equal at all points in a circuit!

$$R_{EQ,ef} = ((R_1 + R_2 + R_3) // (R_4 + R_5) // (R_6 + R_7) + R_8) // R_9 \quad (2.6)$$

Finally, we will consider the case of equivalent resistance between nodes a and e. This is depicted schematically in figure 2.22.

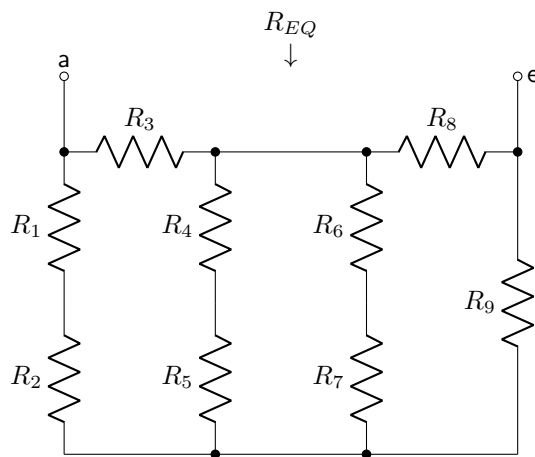


Figure 2.21: Schematic depiction of the equivalent resistance between nodes a and e in the circuit given in figure 2.18.

In this case, resistors R_4 and R_5 can be combined in series, as can R_6 and R_7 . Those equivalent resistances can then be combined in parallel. In addition, R_1 and R_2 can be combined in series. The resulting circuit is shown in figure 2.22.

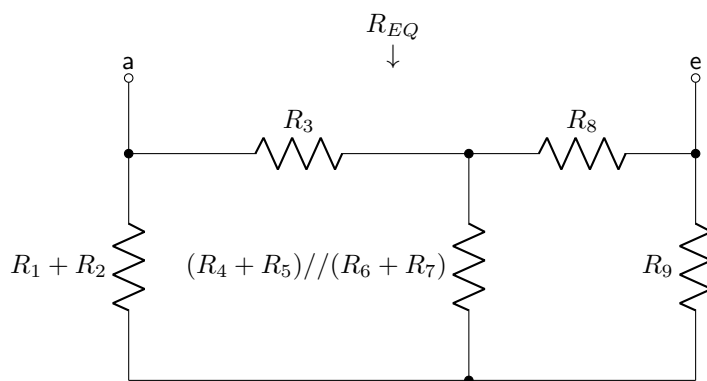


Figure 2.22: Reduced circuit from figure 2.18 when measuring between nodes a and e.

Note that none of the remaining resistors can be combined in either series or in parallel! At this point, we are unable to move forward with the resistance calculation between nodes a and e. Eventually, we will solve this problem with delta-wye and wye-delta transforms (explained in section 2.3). It is crucial to understand that **not all resistors are necessarily going to be connected in parallel or series combinations.**

2.3 Delta-Wye and Wye-Delta Transforms

As mentioned previously, there are situations when a resistive circuit cannot be reduced using parallel and series combinations. In those cases, it may be possible to transform the resistor configuration to a functionally identical equivalent. After completing a transformation, series and parallel combinations may be possible. The two transformations that will be discussed in this book are delta-wye and wye-delta transforms.

A delta circuit arrangement is shown in figure 2.23.

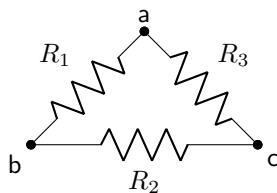


Figure 2.23: A delta circuit.

The resistance between any two nodes in a delta circuit are parallel and series combinations of those resistors, shown in equations 2.7–2.9.

$$R_{ab} = R_1 // (R_2 + R_3) \quad (2.7)$$

$$R_{bc} = R_2 // (R_1 + R_3) \quad (2.8)$$

$$R_{ac} = R_3 // (R_1 + R_2) \quad (2.9)$$

A wye circuit arrangement is shown in figure 2.24. Note that it has one more node than a delta circuit.

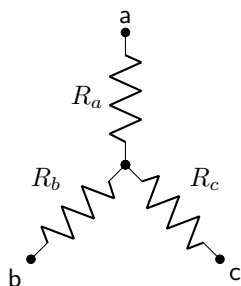


Figure 2.24: A wye circuit.

The resistance between any two nodes in a wye circuit are series combinations of those resistors, shown

in equations 2.10–2.12.

$$R_{ab} = R_a + R_b \quad (2.10)$$

$$R_{bc} = R_b + R_c \quad (2.11)$$

$$R_{ac} = R_a + R_c \quad (2.12)$$

2.3.1 Delta-Wye Transforms

A delta-wye transform modifies a delta arrangement of resistors (figure 2.23) and turns it into an equivalent wye arrangement (figure 2.24). The equivalent resistance between any two nodes of the wye circuit must be identical to the corresponding equivalent resistances of the delta circuit in order to make this transformation valid. Equation 2.7 is set equal to equation 2.10; equation 2.8 is set equal to equation 2.11; and equation 2.9 is set equal to equation 2.12. The terms R_a , R_b , and R_c are solved for. (The algebraic details of these derivations will not be shown in this book.) The results are given in equations 2.13–2.15. These equations govern the delta-wye transform.

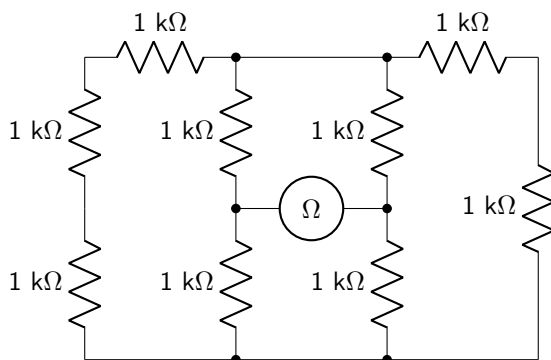
$$R_a = \frac{R_1 R_3}{R_1 + R_2 + R_3} \quad (2.13)$$

$$R_b = \frac{R_1 R_2}{R_1 + R_2 + R_3} \quad (2.14)$$

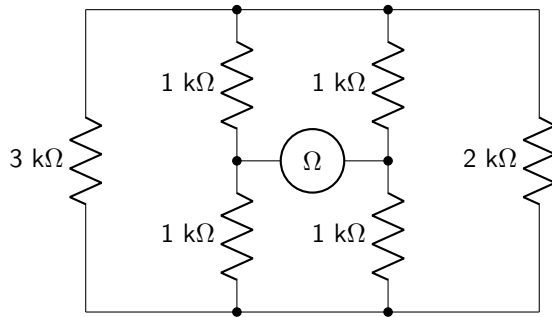
$$R_c = \frac{R_2 R_3}{R_1 + R_2 + R_3} \quad (2.15)$$

Example: Using a delta-wye transform to calculate equivalent resistance

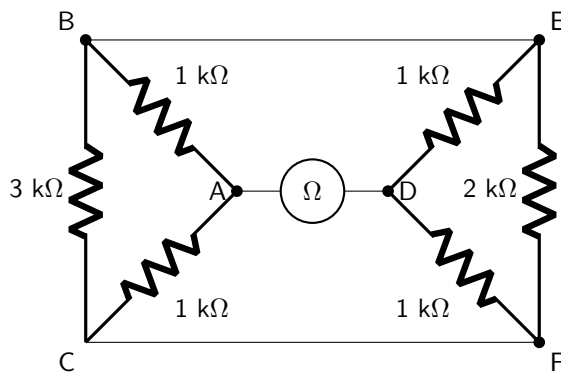
Calculate the equivalent resistance of the circuit shown in figure 2.18 between nodes c and d. Each of the resistors has a value of 1 k Ω . The circuit has been re-drawn below.



Combine all series resistors.



No more resistors that can be combined in either series or in parallel. Two delta-wye transforms can be done. To make it more obvious, the resistors have been slightly re-drawn. Each delta configuration is depicted with thick black outlines, and each node is labeled.



Calculate each of the wye resistor values using equations 2.13–2.15.

$$R_a = \frac{(1 \text{ k}\Omega)(1 \text{ k}\Omega)}{1 \text{ k}\Omega + 3 \text{ k}\Omega + 1 \text{ k}\Omega} = 200 \text{ }\Omega$$

$$R_b = \frac{(1 \text{ k}\Omega)(3 \text{ k}\Omega)}{1 \text{ k}\Omega + 3 \text{ k}\Omega + 1 \text{ k}\Omega} = 600 \text{ }\Omega$$

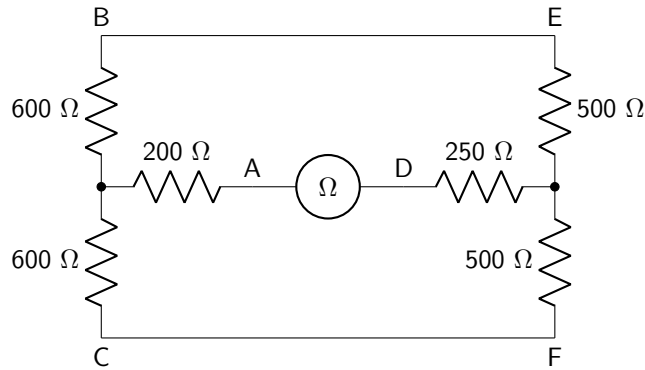
$$R_c = \frac{(3 \text{ k}\Omega)(1 \text{ k}\Omega)}{1 \text{ k}\Omega + 3 \text{ k}\Omega + 1 \text{ k}\Omega} = 600 \text{ }\Omega$$

$$R_d = \frac{(1 \text{ k}\Omega)(1 \text{ k}\Omega)}{1 \text{ k}\Omega + 2 \text{ k}\Omega + 1 \text{ k}\Omega} = 250 \text{ }\Omega$$

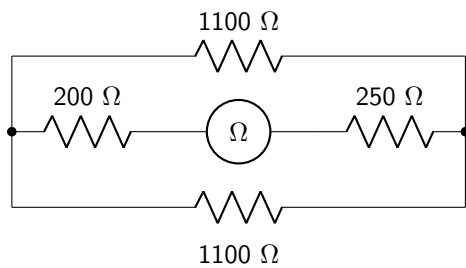
$$R_e = \frac{(1 \text{ k}\Omega)(2 \text{ k}\Omega)}{1 \text{ k}\Omega + 2 \text{ k}\Omega + 1 \text{ k}\Omega} = 500 \text{ }\Omega$$

$$R_f = \frac{(2 \text{ k}\Omega)(1 \text{ k}\Omega)}{1 \text{ k}\Omega + 2 \text{ k}\Omega + 1 \text{ k}\Omega} = 500 \text{ }\Omega$$

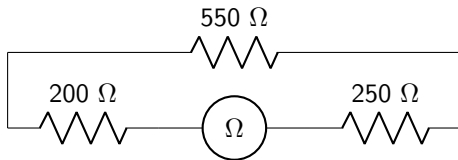
Re-draw the circuit with the wye configurations.



Combine all series resistors. (Both of them are combinations of 500 Ω and 600 Ω .)



Combine all parallel resistors.



The equivalent resistance is equal to the series combination of the remaining resistances.

$$R_{EQ,cd} = 200 \Omega + 550 \Omega + 250 \Omega = 1000 \Omega$$

2.3.2 Wye-Delta Transforms

A wye-delta transform modifies a wye arrangement of resistors (figure 2.24) and turns it into a delta arrangement (figure 2.23). Just as with the delta-wye transforms, the circuits must be equivalent to each other. Equation 2.7 is set equal to equation 2.10; equation 2.8 is set equal to equation 2.11; and equation 2.9 is set equal to equation 2.12. The terms R_1 , R_2 , and R_3 are solved for. (The algebraic details of these derivations will not be shown in this book.) The results are given in equations 2.16–2.18. These equations govern the

wye-delta transform.

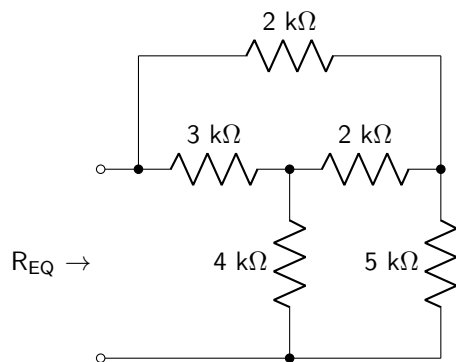
$$R_1 = \frac{R_a R_b + R_b R_c + R_a R_c}{R_c} \quad (2.16)$$

$$R_2 = \frac{R_a R_b + R_b R_c + R_a R_c}{R_a} \quad (2.17)$$

$$R_3 = \frac{R_a R_b + R_b R_c + R_a R_c}{R_b} \quad (2.18)$$

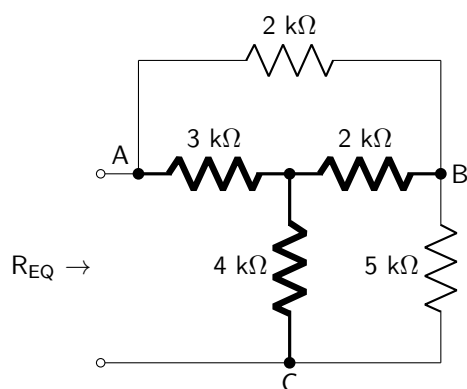
Example: Using a wye-delta transform to calculate equivalent resistance

Calculate the equivalent resistance of the circuit shown.



None of these resistors shares a branch, therefore none are in series. None of these resistors shares two nodes, therefore none are in parallel. A wye-delta transform is required. (Alternatively, a delta-wye transform could be performed.)

The resistors in the wye-arrangement are shown with thick black lines. Each of the nodes is labeled.



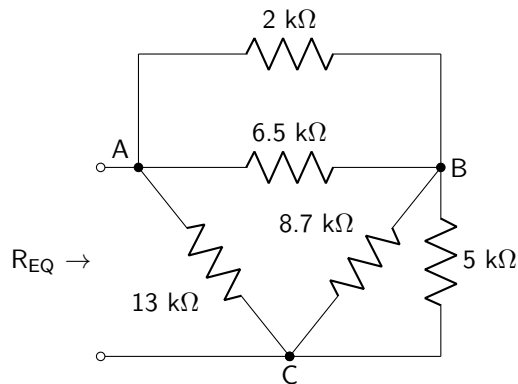
Use equations 2.16–2.18 to solve for R_1 , R_2 , and R_3 .

$$R_1 = \frac{26000 \Omega^2}{4000 \Omega} = 6500 \Omega$$

$$R_2 = \frac{26000 \Omega^2}{3000 \Omega} = 8667 \Omega$$

$$R_3 = \frac{26000 \Omega^2}{2000 \Omega} = 13000 \Omega$$

Re-draw the circuit.



Calculate the equivalent resistance.

$$\begin{aligned} R_{EQ} &= ((2 \text{ k}\Omega // 6.5 \text{ k}\Omega) + (5 \text{ k}\Omega // 8.67 \text{ k}\Omega)) // 13 \text{ k}\Omega \\ &= (1.5 \text{ k}\Omega + 3.2 \text{ k}\Omega) // 13 \text{ k}\Omega \\ &= 4.7 \text{ k}\Omega // 13 \text{ k}\Omega \\ &= 3.45 \text{ k}\Omega \end{aligned}$$

2.4 Ohm's Law

Ohm's law states the relationship between voltage, current, and resistance in a circuit. Current is proportional to voltage drop, and inversely proportional to the resistance of the circuit. In equation form, Ohm's law is usually stated in one of three ways, shown in equations 2.19–2.21. Ohm's law is the first tool that we can use to analyze currents through and voltages dropped over various elements in a circuit.

$$V = IR \quad (2.19)$$

$$I = \frac{V}{R} \quad (2.20)$$

$$R = \frac{V}{I} \quad (2.21)$$

It is important to recall that voltage is a relative measurement (leading to a positive or negative result,

depending on which way it is referenced) and that the sign of current relates to the direction. This means that it is important to pay attention to references and directions when using Ohm's law.

It is vital to pay attention to the units used in Ohm's law. One ohm (the unit for resistance) is equal to one volt divided by one amp. If using units of ohms, all voltages must be expressed in volts and all currents must be expressed in amps. Frequently, resistance will be quantified in units of kilohms. If using units of kilohms, all voltages must be expressed in volts and all currents must be expressed in milliamps.

2.4.1 Calculating Voltage from Current and Resistance

If current and resistance are known, then the voltage drop can be calculated. Consider the resistor shown in figure 2.25. The value of the resistance has no polarity or direction. If the direction of current flow is from node a to node b, then node a corresponds to high potential and node b corresponds to low potential. In that case, V_{ab} (the electric potential at a minus the electric potential at b) is positive. If the direction of current flow is from node b to node a, then node b corresponds to high potential and V_{ab} would be negative.



Figure 2.25: A resistor used to demonstrate Ohm's law.

In other words, current flows from high potential to low potential. The direction of the current can therefore be used to distinguish the high potential end of the circuit element (depicted with a + sign) from the low potential end (depicted with a - sign). This is depicted schematically in figure 2.26.

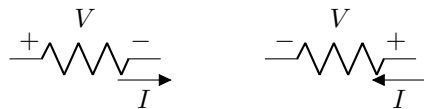
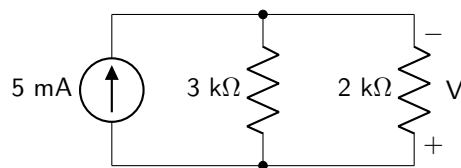


Figure 2.26: Current flows from high to low potential. Therefore, the direction of current can be used to determine which end of a circuit element corresponds to high potential and which corresponds to low potential.

Example: Calculating voltage from current and resistance

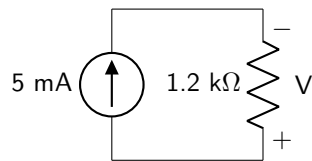
Calculate the voltage drop across the resistors.



Calculate the equivalent resistance of the two resistors.

$$R_{EQ} = \frac{(3 \text{ k}\Omega)(2 \text{ k}\Omega)}{2 \text{ k}\Omega + 3 \text{ k}\Omega} = 1.2 \text{ k}\Omega$$

The circuit can be re-drawn.



The voltage is defined in the opposite direction as the current flow.

$$V = (-5 \text{ mA})(1.2 \text{ k}\Omega) = -6 \text{ V}$$

If the voltage had been defined in the other direction, then it would be measured in the same direction as the current, and the sign would be positive.

2.4.2 Calculating Current from Voltage and Resistance

If the voltage drop over a circuit element as well as the resistance is known, then the current flow can be calculated. Current flows from high potential to low potential. However, current can be assigned in any arbitrary direction; if the assignment of the current flow was from low to high potential, then a negative current will be calculated, as discussed in section 1.3.1 in this book. Therefore, when calculating current in a circuit (as we will do in section 2.7.1), it is not important which direction is chosen to define current, as the real direction will make itself known with the sign of the result. This relationship between voltage drop and current direction assignment is depicted schematically in figure 2.27.

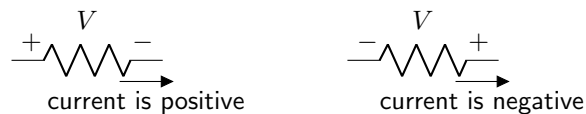
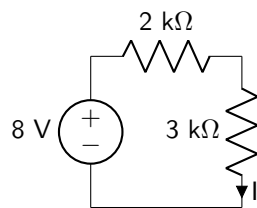


Figure 2.27: Current flows from high to low potential. If the direction of current flow was chosen in this direction, current will be positive (left). If the flow was assigned in the opposite direction, current will be negative (right).

Example: Calculating current from voltage and resistance

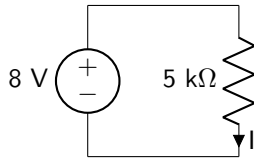
Calculate the current flowing through the circuit.



Calculate the equivalent resistance of the two resistors.

$$R_{EQ} = 2 \text{ k}\Omega + 3 \text{ k}\Omega = 5 \text{ k}\Omega$$

The circuit can be re-drawn.



The current is flowing from high to low potential, so the sign (direction) of the current will be positive.

$$I = \frac{8 \text{ V}}{5 \text{ k}\Omega} = 1.6 \text{ mA}$$

If the current had been defined in the other direction, then it would be measured from low to high potential and the sign would be negative.

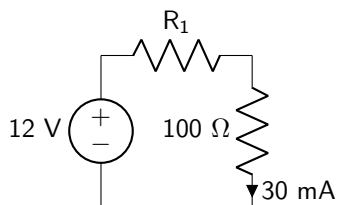
2.4.3 Calculating Resistance from Voltage and Current

When using voltage and current to calculate resistance, it is important to ensure that the directionality of the current and the polarity of the voltage align with each other. In other words: if voltage is measured from high to low potential, then the current flow must be assigned to flow in that direction (from high to low). If the voltage was measured from low to high potential, then the current flow must be assigned to flow from low to high.

In this textbook, the circuits that will be analyzed will all have a positive resistance. This is helpful in ensuring that our polarities/directions are assigned correctly, because we will find that we are always multiplying a positive voltage with a positive current or a negative voltage with a negative current, but never a positive with a negative. However, it is important to note that there are negative resistance circuits which may be encountered in a higher level class. In those cases, it will be crucial to pay attention to the assigned polarities and directions!

Example: Calculating resistance from voltage and current

Calculate the value of resistor R_1 .



Use Ohm's law (equation 2.21) to solve this problem. The total resistance of this circuit is the series combination of R_1 and the $100\ \Omega$ resistor.

Voltage and current are both defined in compatible directions, so none of the signs need to be changed in the calculation. Because ohms are used in the calculation, the current flow through the circuit will be expressed in units of amps. (Alternatively, the resistance could be converted to $k\Omega$ while leaving the current in mA.)

$$\begin{aligned} R_1 + 100\ \Omega &= \frac{12\ \text{V}}{0.03\ \text{A}} \\ R_1 &= 400\ \Omega - 100\ \Omega \\ &= 300\ \Omega \end{aligned}$$

2.4.4 Ohm's Law and the Power Equation

Together with the power equation, it is possible to know any two of four properties (current, voltage, resistance, and power) and solve for the others. The equations that link Ohm's law and the power equation are shown in equations 2.22–2.24.

$$P = IV \tag{2.22}$$

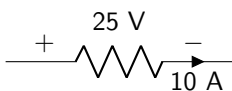
$$P = I^2R \tag{2.23}$$

$$P = \frac{V^2}{R} \tag{2.24}$$

Just as we were concerned with units while utilizing Ohm's law, we are similarly interested in paying close attention to units when using the power equations. If ohms are used in equations 2.22–2.24, then voltage must be expressed in volts, current must be expressed in amps, and the power units will be expressed in watts. Alternatively, if kilohms are used for resistance, then voltage must be expressed in volts, current must be expressed in milliamperes, and power will be expressed in milliwatts.

Example: Power calculation

Calculate the power consumed by the resistor in the circuit. Then, calculate the value of the resistor.



Use equation 2.22 to calculate the power consumed by the resistor.

$$P = (10\ \text{A})(25\ \text{V}) = 250\ \text{W}$$

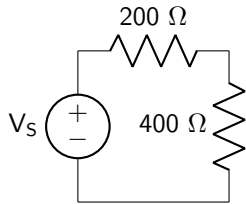
Use Ohm's law (equation 2.21) to calculate the value of the resistor. (Alternatively, equation 2.19 or 2.20

could be used to solve for R.)

$$R = \frac{25 \text{ V}}{10 \text{ A}} = 2.5 \Omega$$

Example: Calculating voltage

The 200Ω resistor is consuming 500 mW of power. Calculate the value of the voltage source.



Calculate the current that flows through the circuit by solving equation 2.23 for I . The units will be converted to V, A, Ω , and W.

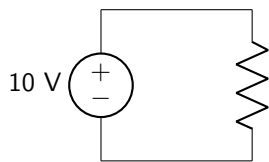
$$I = \sqrt{\frac{P}{R}} = \sqrt{\frac{0.5 \text{ W}}{200 \Omega}} = 0.05 \text{ A}$$

Use Ohm's law (equation 2.19) to calculate the value of the voltage source that is required to generate this current flow.

$$V_S = (0.05 \text{ A})(200 \Omega + 400 \Omega) = 30 \text{ V}$$

Example: Calculating minimum safe resistor value

What is the minimum resistor value that can be used in this circuit, if the resistor is rated for a maximum power of $1/4 \text{ W}$?



Use equation 2.24.

$$R_{min} = \frac{(10 \text{ V})^2}{0.25 \text{ W}} = 400 \Omega$$

2.5 Voltage Divider

The voltage divider is a tool that uses Ohm's law to calculate the voltage dropped over resistive elements that are in series with each other without having to calculate current.

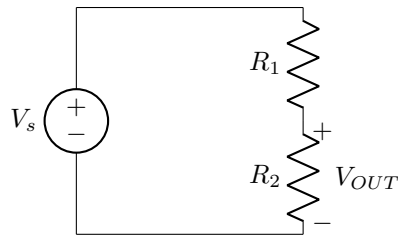


Figure 2.28: Schematic used to demonstrate the voltage divider tool.

Given a circuit with a voltage source and two resistors in series (shown in figure 2.28), Ohm's law can be used to calculate the overall voltage/current relationship of the circuit, and is shown in equation 2.25.

$$I = \frac{V_S}{R_1 + R_2} \quad (2.25)$$

Because the current flowing through the resistor of interest (R_2 in this example) is equal to the current flowing through the entire circuit, we can use Ohm's law to calculate the voltage drop over the resistor ($V_{OUT} = IR_2$), and plug in the result of equation 2.25 to remove the current term from the equation. This is shown in equation 2.26.

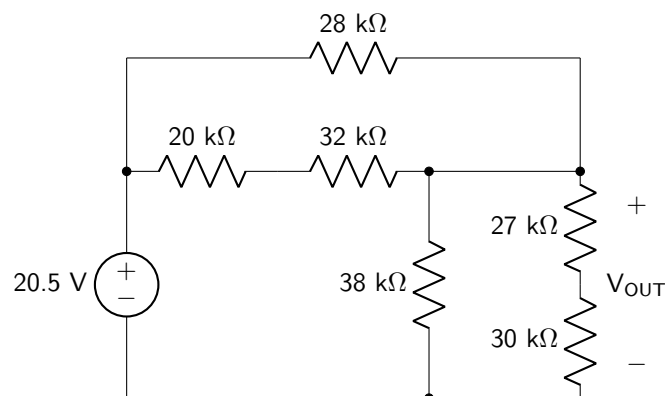
$$V_{OUT} = \frac{V_S R_2}{R_1 + R_2} \quad (2.26)$$

In general, equation 2.27 can be used to find the voltage dropped over the k^{th} resistor of n resistors in series (where $k < n$), given a source voltage of V_S . As mentioned, there is no need to calculate current as long as the voltage and resistor values are known.

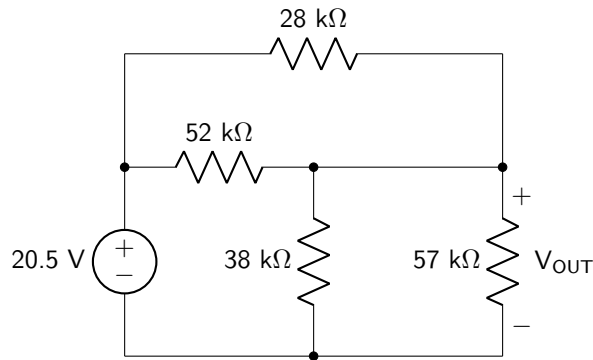
$$V_k = V_S \left(\frac{R_k}{R_1 + R_2 + R_3 + \dots + R_k + \dots + R_n} \right) \quad (2.27)$$

Example: Using the voltage divider

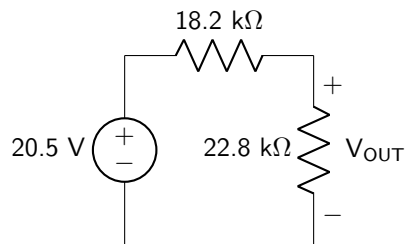
Calculate V_{OUT} in the following circuit.



Combine all resistors that are in series and re-draw the circuit.



The 28 kΩ and 52 kΩ resistors are in parallel with each other and have an equivalent resistance of 18.2 kΩ. The 38 kΩ and 57 kΩ resistors are in parallel with each other and have an equivalent resistance of 22.8 kΩ. The circuit can be re-drawn again.



Use the voltage divider rule to calculate the output voltage.

$$V_{OUT} = 20.5 \text{ V} \left(\frac{22.8 \text{ k}\Omega}{22.8 \text{ k}\Omega + 18.2 \text{ k}\Omega} \right) = 11.4 \text{ V}$$

2.5.1 Wheatstone Bridge

A Wheatstone bridge is a measuring tool that can be used to indirectly determine the resistance of an object with otherwise unknown resistance. It is useful when measuring small resistances (or changes in resistance) that may not be within the accuracy of an ohmmeter. As will be demonstrated, the unknown resistance value is largely invariant to noise from the voltage source as its value is independent from the source. The schematic of a Wheatstone bridge is shown in figure 2.29, with the unknown resistance labeled R_X .

The value of R_X can be determined if the bridge is balanced. This occurs when the value of the variable resistor (R_2) is changed such that the voltmeter reads 0 V. When the bridge is balanced, V_1 and V_2 are equal. The voltage divider rule can be used to calculate V_1 and V_2 . Then, set V_1 and V_2 equal to calculate R_X . This is shown in equation 2.28 and derived below.

$$R_X = \frac{R_2 R_3}{R_1} \quad (2.28)$$

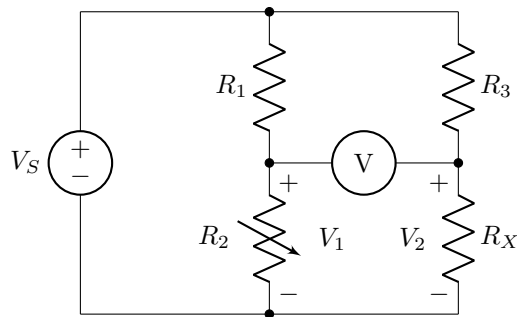


Figure 2.29: Circuit schematic of a Wheatstone bridge circuit. R_2 is a variable resistor and R_X is unknown.

DERIVATION

$$\begin{aligned}
 V_S \left(\frac{R_2}{R_1 + R_2} \right) &= V_S \left(\frac{R_X}{R_3 + R_X} \right) \\
 \frac{R_2}{R_1 + R_2} &= \frac{R_X}{R_3 + R_X} \\
 R_2(R_3 + R_X) &= R_X(R_1 + R_2) \\
 R_2R_3 + R_2R_X &= R_XR_1 + R_XR_2 \\
 R_2R_3 &= R_XR_1 \\
 R_X &= \frac{R_2R_3}{R_1}
 \end{aligned}$$

2.6 Current Divider

The current divider uses Ohm's law to calculate the current flowing through resistive elements that are in parallel with each other without having to calculate voltages.

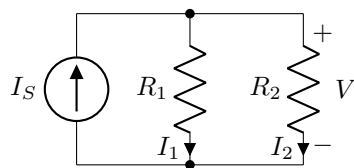


Figure 2.30: Schematic used to demonstrate the current divider tool.

Given a circuit with a current source and two resistors in parallel (shown in figure 2.30), Ohm's law can be used to calculate the overall voltage/current relationship of this circuit, and is shown in equation 2.29.

$$V = I_S \left(\frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} \right) \quad (2.29)$$

Because the voltage dropped over both resistors is equal to V , Ohm's law applied over each individual resistor can now be used to determine the individual currents flowing through each resistor. Ohm's law

states that the current flowing through R_1 is equal to V/R_1 . Therefore, equation 2.29 can be divided by R_1 to calculate the current flowing through that resistor. This is shown in equation 2.30.

$$I_1 = I_S \left(\frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2}} \right) \quad (2.30)$$

Note the similarity with the voltage divider rule. Here we are using reciprocal resistances due to the parallel configuration of the resistors. In general, equation 2.31 (derived below) can be used to find the current flowing through the k^{th} resistor of n resistors in parallel (where $k < n$), given a source current of I_S . As mentioned, there is no need to calculate the total voltage as long as the current and resistor values are known.

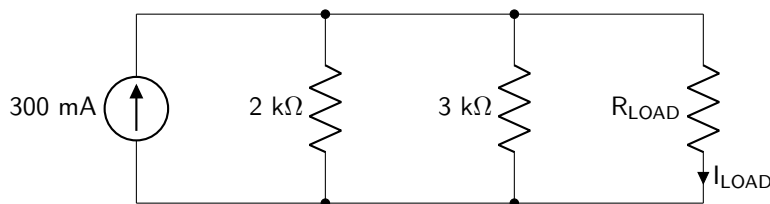
$$I_k = I_S \left(\frac{R_{EQ}}{R_k} \right) \quad (2.31)$$

DERIVATION

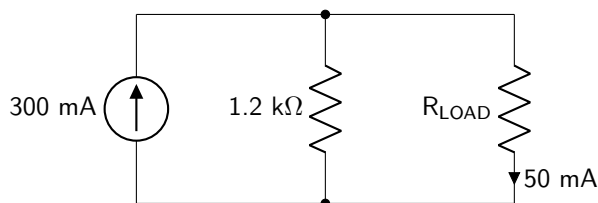
$$\begin{aligned} I_k &= I_S \left(\frac{\frac{1}{R_k}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots + \frac{1}{R_k} + \dots + \frac{1}{R_n}} \right) \\ &= I_S \left(\frac{\frac{1}{R_k}}{\frac{1}{R_{EQ}}} \right) \\ &= I_S \left(\frac{R_{EQ}}{R_k} \right) \end{aligned}$$

Example: Using the current divider

The current through the load resistor must be 50 mA. Calculate the value of the load resistance that is required to accomplish this.



Reduce the circuit by combining the 2 kΩ and 3 kΩ resistors in parallel.



Use the current divider rule to calculate the load resistance.

$$50 \text{ mA} = 300 \text{ mA} \left(\frac{(1.2 \text{ k}\Omega)(R_{LOAD})}{(1.2 \text{ k}\Omega + R_{LOAD})} \right)$$

$$\frac{50 \text{ mA}}{300 \text{ mA}} = \left(\frac{1.2 \text{ k}\Omega}{1.2 \text{ k}\Omega + R_{LOAD}} \right)$$

$$1.2 \text{ k}\Omega = 0.167(1.2 \text{ k}\Omega + R_{LOAD})$$

$$7.2 \text{ k}\Omega = 1.2 \text{ k}\Omega + R_{LOAD}$$

$$R_{LOAD} = 6 \text{ k}\Omega$$

2.7 Kirchhoff's Laws

Kirchhoff's laws, used together with Ohm's law, form a powerful set of tools for analyzing the currents through and voltages dropped over elements in even very complicated circuits. The two Kirchhoff's laws are Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL).

2.7.1 Kirchhoff's Current Law (KCL)

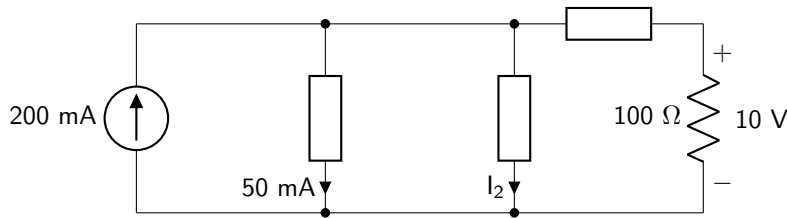
Kirchhoff's current law (KCL) is a byproduct of the principle of conservation of charge. At any point in a circuit, the charge flowing in to that point must be equal to the charge flowing out of that point. This relates to current because current is equal to the change in charge over time. If net charge is conserved, then net current must be as well. KCL can be stated in three slightly different, but identical, ways.

- The sum of all current entering a node is equal to zero.
- The sum of all current leaving a node is equal to zero.
- The sum of all current entering a node is equal to the sum of all current leaving a node.

An important consequence of KCL is that the current flowing through any branch of a circuit is the same everywhere throughout that branch at any moment in time. The currents that are calculated using KCL are called branch currents. (This may seem like the only type of current that can be measured, but we will learn later in this chapter that mesh analysis is a tool that enables us to find mesh currents, which are different from branch currents.)

Example: Kirchhoff's current law

Calculate the value of I_2 .



KCL states that the sum of all currents entering the node connecting all three generic elements to the high potential end of the current source must be zero. Use this and Ohm's law to solve for I_2 .

$$0 = 200 \text{ mA} - 50 \text{ mA} - I_2 - \frac{10 \text{ V}}{0.1 \text{ k}\Omega}$$

$$I_2 = 50 \text{ mA}$$

2.7.2 Kirchhoff's Voltage Law (KVL)

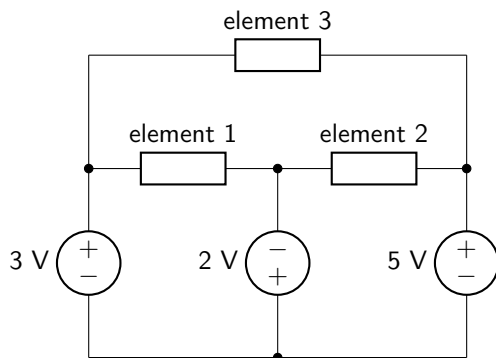
Kirchhoff's voltage law (KVL) is a byproduct of the principle of conservation of energy. Energy must be conserved in each loop in a circuit. Energy is equal to charge times voltage, and as energy and charge are both conserved, voltage must be as well. KVL can be stated in three slightly different, but identical, ways.

- The sum of all of the voltage drops in a loop is equal to zero.
- The sum of all of the voltage rises in a loop is equal to zero.
- The sum of all of the voltage drops in a loop is equal to the sum of all of the voltage rises in a loop.

An important consequence of KVL is that the voltage drops over parallel elements are identical.

Example: Kirchhoff's voltage law

Determine the value of the voltage drop over each circuit element and indicate the correct polarity.



KVL states that the sum of all voltage drops in a loop must be equal to zero. The lower left loop can

be analyzed in a clockwise fashion to calculate the voltage drop over element 1.

$$0 = -3 \text{ V} + V_1 - 2 \text{ V}$$

$$V_1 = 5 \text{ V}$$

The lower right loop can be analyzed in a clockwise fashion to calculate the voltage drop over element 2.

2. The negative sign indicates that the right-hand side of the element is the high potential side.

$$0 = 2 \text{ V} + V_2 + 5 \text{ V}$$

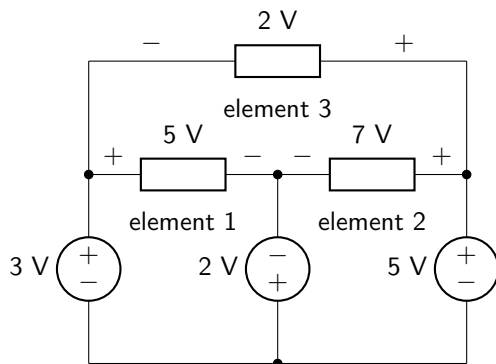
$$V_2 = -7 \text{ V}$$

The loop consisting of the 3 V source, element 3, and the 5 V source can be analyzed in a clockwise fashion to calculate the voltage drop over element 3. The negative sign indicates that the right-hand side of the element is the high potential side.

$$0 = -3 \text{ V} + V_3 + 5 \text{ V}$$

$$V_3 = -2 \text{ V}$$

The circuit can now be re-drawn, indicating the voltage and polarity of each circuit element.



2.7.3 Finding Matrix Equations from KCL and KVL

Together, KCL and KVL can be used to solve for every branch current in a circuit. Those branch currents can be multiplied by resistance to find voltage drops over each resistor in a circuit. By analyzing nodes and loops in a circuit, a number of equations (using KCL, KVL, and Ohm's law) can be found. If there are n unknown currents in the circuit, n linearly independent equations will be needed. A matrix can then be used to solve for the unknowns.

This book will not discuss matrix reduction techniques, which is better suited for a linear algebra textbook. Gauss-Jordan elimination can be used to find the reduced row echelon form of a matrix. Alternatively,

a graphing calculator can be used to solve the matrices. In short: this book is not about how to solve matrix equations. This book is about how to set up those matrix equations by using KCL and KVL.

Consider the circuit in figure 2.31. There are four unknown currents so four linearly independent equations will need to be used to solve for each current. The direction of each current has been arbitrarily chosen.

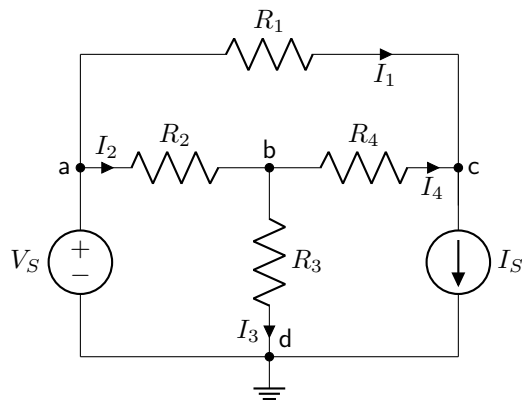


Figure 2.31: Circuit used to demonstrate using KCL and KVL to solve for unknown currents.

To find a sufficient number of linearly independent equations, analysis can occur at nodes (using KCL) and around loops (using KVL). It is important to try to find nodes at which we know (or want to know) all of the currents entering and exiting. In figure 2.31, there are four nodes, each labeled with a letter. This means we could generate four KCL equations (one at each node). However, by closely analyzing nodes a and d, we see that the current flowing through the voltage source is unknown. Therefore, it would not be ideal to include nodes a or d in our set of equations. While not a standard term in electrical engineering, this textbook will use the term *perfect node* to denote a node at which we know or care about all of the currents entering or exiting that node.

The circuit in figure 2.31 therefore has two perfect nodes: node b and node c. These nodes will be used solve KCL, giving us our first two linearly independent equations, shown in equations 2.32 and 2.33.

$$I_2 - I_3 - I_4 = 0 \quad (2.32)$$

$$I_1 + I_4 - I_S = 0 \quad (2.33)$$

We turn to KVL to find the remaining two equations. Just as we searched for perfect nodes to find KCL equations, we will look for *perfect loops* to find KVL equations. In a perfect loop, all of the voltages are known or are known resistances multiplied by currents we hope to find. In figure 2.31, any loop that contains the current source would not be a perfect loop; we do not know (or care to find) the voltage dropped over the current source.

It is also of the utmost important to find linearly independent equations when using KVL. This means that the loops that we choose must contain at least one unique circuit element from the others. For example (which is not necessarily relevant to the solution of this particular circuit): the loop containing V_S , R_2 , and

R_3 as well as the loop containing R_3 , R_4 , and I_S are linearly independent. Including a third loop of V_S , R_2 , R_4 , and I_S would not add anything new to the circuit. The equations derived from these three loops would not lead to three linearly independent equations; they would only lead to two linearly independent equations.

Turning back to the example at hand: we desire two linearly independent equations that come from perfect loops. Two loops that satisfy those criteria are: the loop containing V_S , R_2 , and R_3 and the loop containing R_1 , R_4 , and R_2 . Apply KVL around each loop. Use the direction of the current to determine if the voltage is a rise or a drop. Voltage rises will be negative, and voltage drops will be positive. The two equations are given in equation 2.34 and 2.35.

$$-V_S + R_2I_2 + R_3I_3 = 0 \quad (2.34)$$

$$R_1I_1 - R_4I_4 - R_2I_2 = 0 \quad (2.35)$$

Now that four linearly independent equations have been found, they can be put into the form of $\alpha I_1 + \beta I_2 + \gamma I_3 + \delta I_4 = c$ and placed into a matrix. Each of the equations, rewritten, are shown in equation 2.36.

$$\begin{array}{cccccc} & I_2 & -I_3 & -I_4 & = & 0 \\ I_1 & & & +I_4 & = & I_S \\ & R_2I_2 & +R_3I_3 & & = & V_S \\ R_1I_1 & -R_2I_2 & & -R_4I_4 & = & 0 \end{array} \quad (2.36)$$

The corresponding matrix is shown in equation 2.37. Now it can be solved to find each of the individual currents.

$$\begin{bmatrix} 0 & 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 & I_S \\ 0 & R_2 & R_3 & 0 & V_S \\ R_1 & -R_2 & 0 & -R_4 & 0 \end{bmatrix} \quad (2.37)$$

One final note about matrix analysis of KCL and KVL. It is not always possible to find a sufficient number of perfect loops and perfect nodes. If there are more unknowns than equations after exhausting all of the perfect nodes and loops, then it is time to define a new unknown. Then another node or loop equation will be available to use.

2.8 Mesh Analysis

Mesh analysis is another tool that uses KCL and KVL to solve for unknowns. In this case, *mesh currents* (as opposed to branch currents, which the previous analysis tools calculated) will be derived. In this book, mesh currents are defined in a clockwise direction.

2.8.1 Branch vs. Mesh Currents

The circuit shown in figure 2.32 has two meshes (and therefore two mesh currents: I_A and I_B). In addition, there are three branch currents defined as I_1 , I_2 , and I_3 .

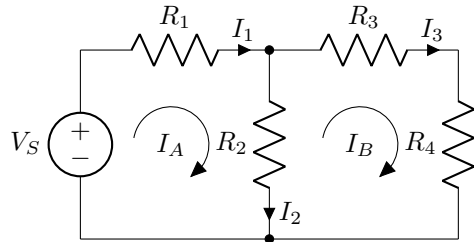


Figure 2.32: This circuit contains two meshes and three defined branches.

Each of the branch currents denote how much current actually flows through each circuit element. This makes the concept of branch currents relatively easy to understand. Branch currents are something we could measure with an ammeter. However, a mesh current is different from a branch current. It is more of an abstraction than a physical parameter. It is possible to relate all of the branch and mesh currents. These relationships are given in equations 2.38–2.40. Note that when a branch current is “shared” between two meshes, then the branch current is equal to the sum of both currents (paying close attention to the direction).

$$I_1 = I_A \quad (2.38)$$

$$I_2 = I_A - I_B \quad (2.39)$$

$$I_3 = I_B \quad (2.40)$$

2.8.2 The Mesh Analysis Method

The steps to performing mesh analysis are given below.

1. Identify each mesh and draw and label each mesh (use a clockwise direction).
2. Perform KVL around each mesh (using the mesh currents) and determine the corresponding equations.
3. If a current source is contained within a mesh, that mesh current will be equal to the value of the current source.
4. If a current source is shared between two meshes:
 - (a) pretend that the current source does not exist (treat it as an open circuit) and find the KVL equation for the loop connecting the two meshes (called a supermesh), then
 - (b) relate the two mesh currents in the supermesh to the current source (branch current) to obtain a new equation.

5. If there is a dependent source (or sources), additional equations relating the controlling values may be required.

The circuit shown in figure 2.33 will be used to demonstrate the mesh analysis procedure. It has been carefully chosen to contain both a dependent source and a supermesh. Each of the meshes have already been identified and labeled.

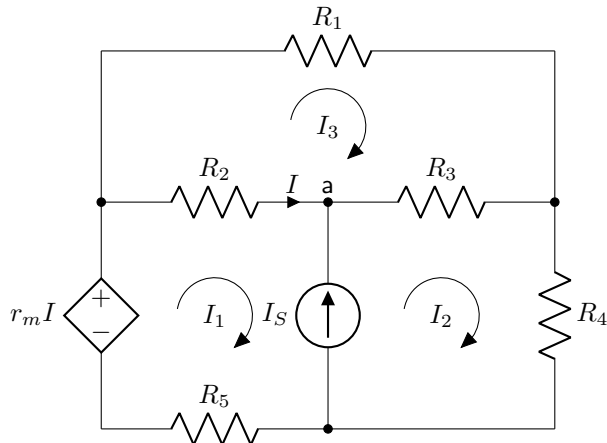


Figure 2.33: Circuit schematic used to demonstrate the mesh analysis method.

There is only one regular mesh in this circuit: the one containing mesh current I_3 . The other two meshes correspond to a supermesh due to the current source that exists at the intersection of the two meshes. Therefore there is only one mesh equation to find, shown in equation 2.41. (Note that we are ignoring I , the branch current used to define the current-controlled voltage source.) This concludes step two.

$$R_1 I_3 + R_3 (I_3 - I_2) + R_2 (I_3 - I_1) = 0 \quad (2.41)$$

Step three does not apply to this circuit. There are no current sources that are unshared between meshes. Step four introduces the supermesh. The supermesh consists of the two loops containing I_1 and I_2 . Because of the existence of the supermesh, KCL will be applied at node a. The supermesh equation is given in equation 2.42 and the KCL equation is shown in equation 2.43.

$$-r_m I + R_2 (I_1 - I_3) + R_3 (I_2 - I_3) + R_4 I_2 + R_5 I_1 = 0 \quad (2.42)$$

$$I_1 + I_S - I_2 = 0 \quad (2.43)$$

At this point, there are three equations. However, the presence of the controlling current (I) provides an additional unknown. This means that one more equation is required. We can use the relationship between branch currents and mesh currents to find an equation for the controlling current, shown in equation 2.44. While this equation can be directly plugged in to the supermesh equation (equation 2.42), it is the opinion

of the author that it eliminates error to include it as a separate equation to be used in the matrix.

$$I = I_1 - I_3 \quad (2.44)$$

Now that there are four linearly independent equations (equations 2.41–2.44), they can be rewritten in the form of $\alpha I_1 + \beta I_2 + \gamma I_3 + \delta I = c$ and placed into a matrix. Each of the equations, rewritten, are shown in equation 2.45.

$$\begin{array}{cccccc} -R_2 I_1 & -R_3 I_2 & +(R_1 + R_2 + R_3) I_3 & & = 0 \\ (R_2 + R_5) I_1 & +(R_3 + R_4) I_2 & -(R_2 + R_3) I_3 & -r_m I & = 0 \\ I_1 & -I_2 & & & = -I_S \\ I_1 & & -I_3 & -I & = 0 \end{array} \quad (2.45)$$

The corresponding matrix is shown in equation 2.46. It can be solved to find each of the mesh currents (as well as the controlling current).

$$\begin{bmatrix} -R_2 & -R_3 & (R_1 + R_2 + R_3) & 0 & 0 \\ (R_2 + R_5) & (R_3 + R_4) & -(R_2 + R_3) & -r_m & 0 \\ 1 & -1 & 0 & 0 & -I_S \\ 1 & 0 & -1 & -1 & 0 \end{bmatrix} \quad (2.46)$$

Example Problems

Equivalent Resistance

1. Calculate the equivalent resistance of the resistors shown in figure 2.34.

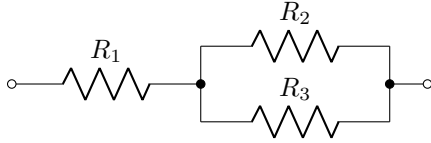


Figure 2.34: Circuit diagram for equivalent resistance question 1.

2. Minimize the circuit diagram shown in figure 2.35 as much as possible.

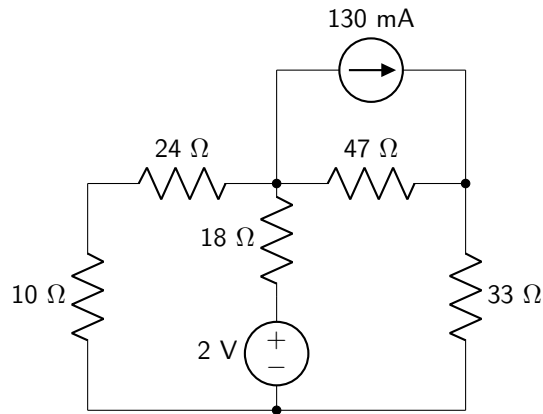


Figure 2.35: Circuit diagram for equivalent resistance question 2.

3. Use the circuit diagram shown in figure 2.36 to calculate the equivalent resistance between nodes a and b.
 - a. Each resistor has a value of $1 \text{ k}\Omega$.

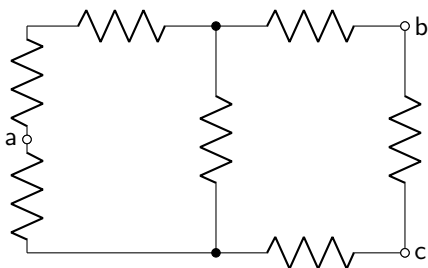


Figure 2.36: Circuit diagram for equivalent resistance questions 3–4.

4. Use the circuit diagram shown in figure 2.36 to calculate the equivalent resistance between nodes b and c. Each resistor has a value of $1 \text{ k}\Omega$.

5. Calculate the equivalent resistance of the resistors shown in figure 2.37.

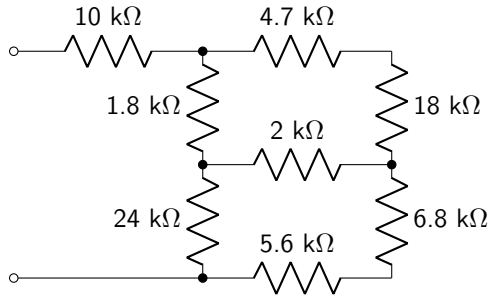


Figure 2.37: Circuit diagram for equivalent resistance question 5.

Ohm's Law

6. If the voltage source can supply a maximum current of 2 A, what is the minimum value of R_X that can be used in the circuit shown in figure 2.38.

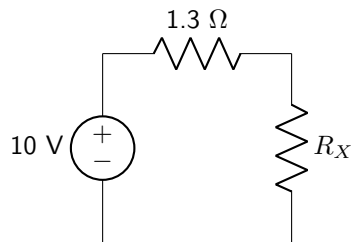


Figure 2.38: Circuit diagram for Ohm's law question 6.

7. Calculate I_X in the circuit shown in figure 2.39.

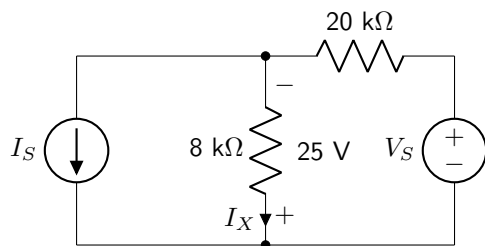


Figure 2.39: Circuit diagram for Ohm's law question 7.

8. Calculate the amount of power consumed by the $20\ \Omega$ resistor in the circuit shown in figure 2.40.

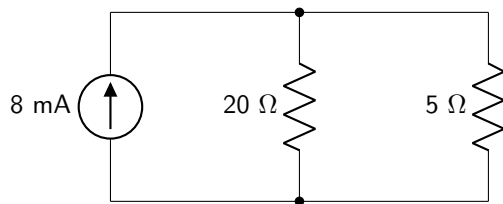


Figure 2.40: Circuit diagram for Ohm's law question 8.

9. Calculate the amount of power supplied by the load in the circuit shown in figure 2.41.

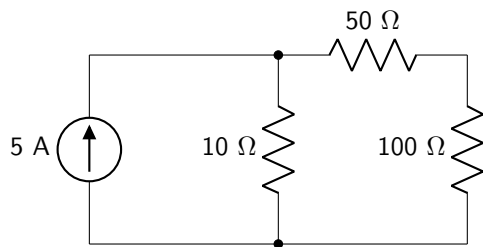


Figure 2.41: Circuit diagram for Ohm's law question 9.

10. Calculate the minimum value of R that can be used to keep the power consumed by either resistor to less than or equal to 250 mW in the circuit shown in figure 2.42.

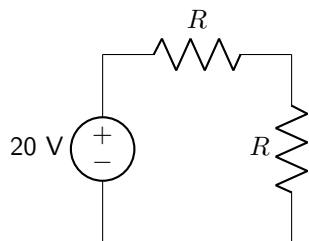


Figure 2.42: Circuit diagram for Ohm's law question 10.

Voltage and Current Divider

11. Use the voltage divider rule to calculate V_X in the circuit shown in figure 2.43.

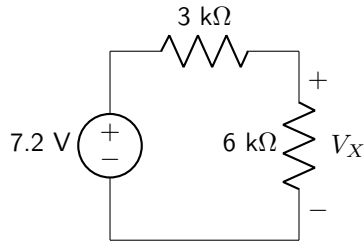


Figure 2.43: Circuit diagram for voltage and current divider question 11.

12. Use the current divider rule to calculate I_X in the circuit shown in figure 2.44.

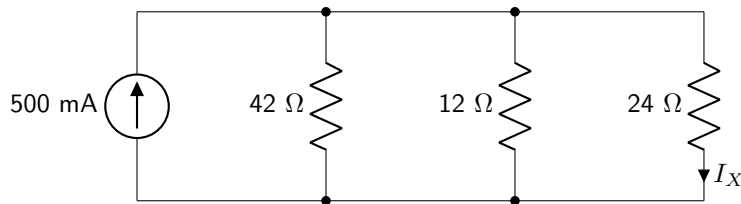


Figure 2.44: Circuit diagram for voltage and current divider question 12.

13. Use the voltage divider rule to calculate V_X in the circuit shown in figure 2.45.

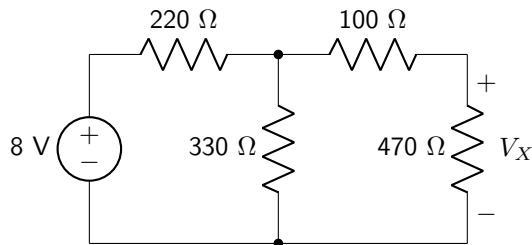


Figure 2.45: Circuit diagram for voltage and current divider question 13.

14. Use the current divider rule to calculate I_X in the circuit shown in figure 2.46.

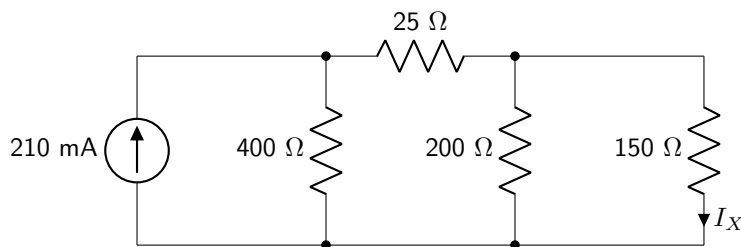


Figure 2.46: Circuit diagram for voltage and current divider question 14.

15. Use the voltage divider rule to calculate V_{X1} and V_{X2} in the circuit shown in figure 2.47.

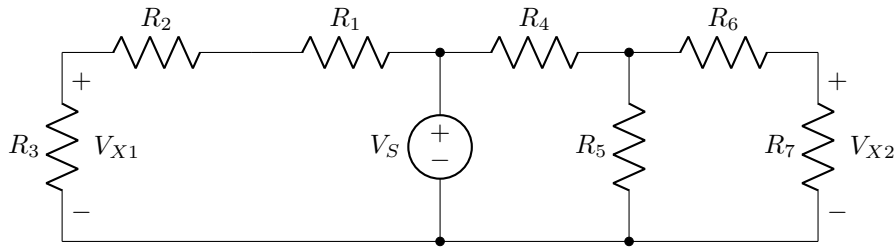


Figure 2.47: Circuit diagram for voltage and current divider question 15.

Kirchhoff's Laws

16. Calculate V_X in the circuit shown in figure 2.48.

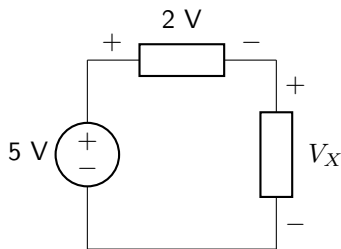


Figure 2.48: Circuit diagram for Kirchhoff's laws question 16.

17. Calculate I_X in the circuit shown in figure 2.49.

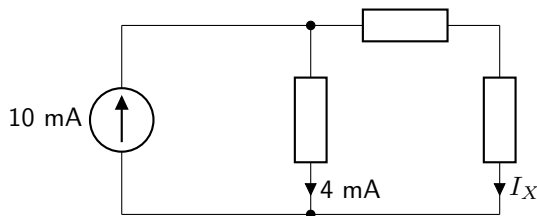


Figure 2.49: Circuit diagram for Kirchhoff's laws question 17.

18. Calculate I_X in the circuit shown in figure 2.50.

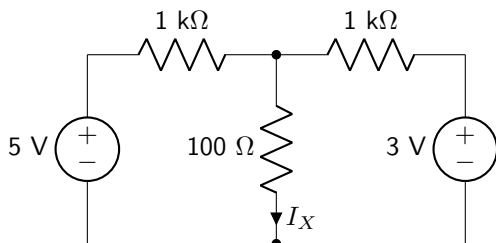


Figure 2.50: Circuit diagram for Kirchhoff's laws question 18.

19. Calculate V_X in the circuit shown in figure 2.51.

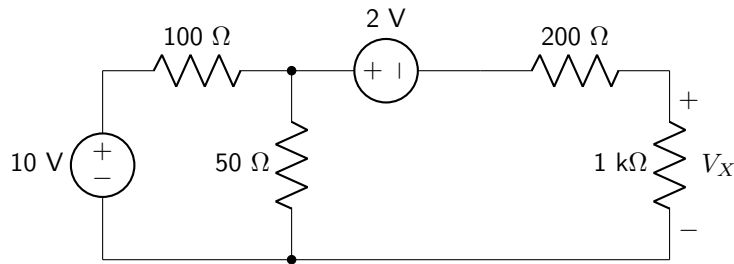


Figure 2.51: Circuit diagram for Kirchhoff's laws question 19.

20. Calculate V_X in the circuit shown in figure 2.52.

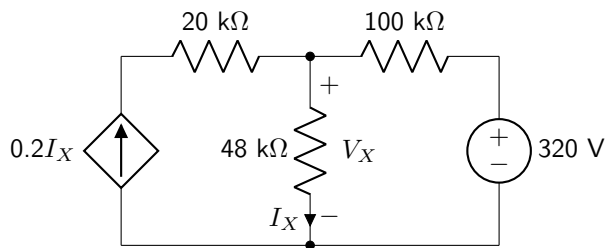


Figure 2.52: Circuit diagram for Kirchhoff's laws question 20.

Mesh Analysis

21. Calculate mesh currents I_A and I_B in the circuit shown in figure 2.53.

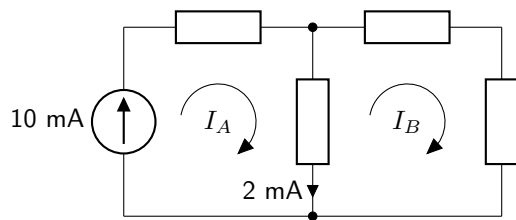


Figure 2.53: Circuit diagram for mesh analysis question 21.

22. A branch is shared by two clockwise meshes. The left mesh current is 3 A and the right mesh current is -6 A. Calculate the branch current.

23. Calculate mesh current I_X in the circuit shown in figure 2.54. Assume that each mesh contains at least one linear circuit element.

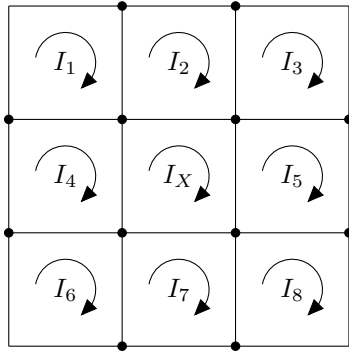


Figure 2.54: Circuit diagram for mesh analysis question 23.

24. Use mesh analysis to calculate V_X in the circuit shown in figure 2.51 (in the Kirchoff's laws section).
25. Use mesh analysis to calculate V_X in the circuit shown in figure 2.55.

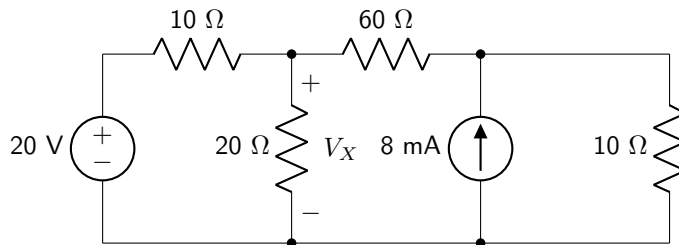


Figure 2.55: Circuit diagram for mesh analysis question 25.

3 Circuit Theorems

In this chapter, various theorems that can be used to analyze circuit properties will be explored. These theorems are all handy tools that can be placed into our circuit analysis toolbox. Between this and the previous chapter, all of the tools and skills required to solve all of the problems in this textbook have been made available. The remaining chapters in this book will use these tools to analyze interesting and complex circuits.

3.1 Superposition

Superposition applies to all linear circuits (which is the only kind of circuit explored in this textbook). If a linear circuit has n independent sources, then n subcircuits can be created, each with one of the independent sources activated and all others deactivated. Add the properties of each of the subcircuits together to find the total value of the property. To deactivate a voltage source, replace the voltage source with a short circuit (corresponding to 0 V). To deactivate a current source, replace the current source with an open circuit (corresponding to 0 A). Any dependent sources that exist in the circuit cannot be eliminated and must be present in all subcircuits.

It is not necessary to perform superposition to determine circuit properties. As discussed in the previous chapter, KCL/KVL and mesh analysis are capable of solving for any unknown current or voltage in a circuit. However, superposition may be simpler and more straightforward to solve, especially in cases with few independent sources.

The circuit shown in figure 3.1 will be used to demonstrate the superposition theorem.

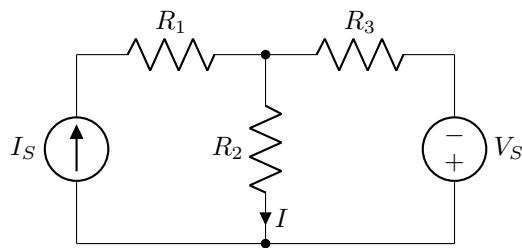


Figure 3.1: Circuit schematic used to demonstrate the superposition theorem.

As there are two independent sources, two subcircuits can be created. The first, shown in figure 3.2 contains the current source. The voltage source is deactivated (replaced with a short circuit).

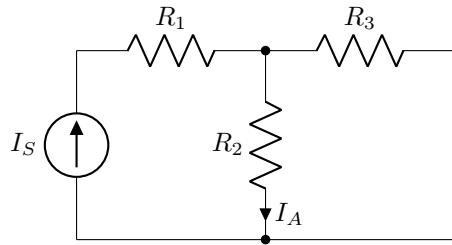


Figure 3.2: A subcircuit of figure 3.1 with the voltage source deactivated.

There are many tools that can be used to find I_A , but the current divider will be used. The results are shown in equation 3.1.

$$I_A = I_S \left(\frac{R_3}{R_2 + R_3} \right) \quad (3.1)$$

The second subcircuit, shown in figure 3.3, contains the voltage source. The current source has been deactivated by replacing it with an open circuit.

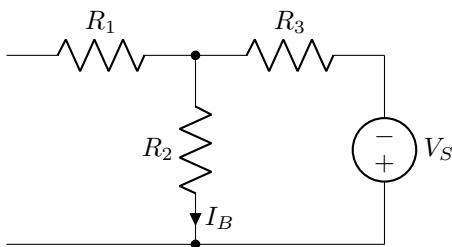


Figure 3.3: A subcircuit of figure 3.1 with the current source deactivated.

Because R_1 is not connected in a complete path, it can be disregarded. Ohm's law is sufficient to calculate the current I_B , shown in equation 3.2.

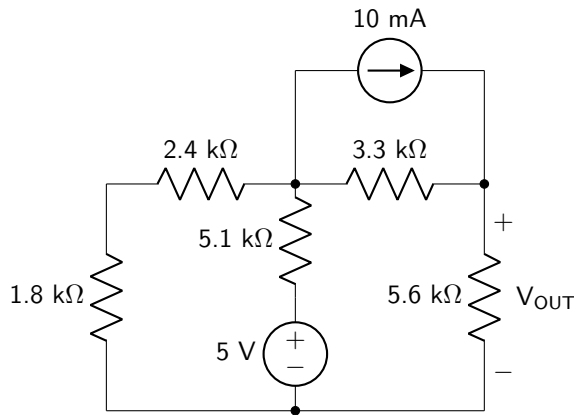
$$I_B = \frac{-V_S}{R_2 + R_3} \quad (3.2)$$

The total current, I , through the original circuit shown in figure 3.1 is equal to the sum of both of the subcircuit currents ($I = I_A + I_B$). This is shown in equation 3.3.

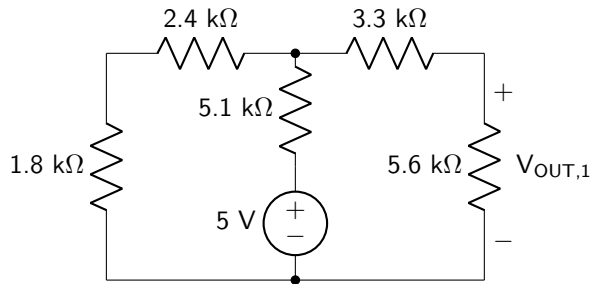
$$I = I_S \left(\frac{R_3}{R_2 + R_3} \right) - \frac{V_S}{R_2 + R_3} \quad (3.3)$$

Example: Superposition

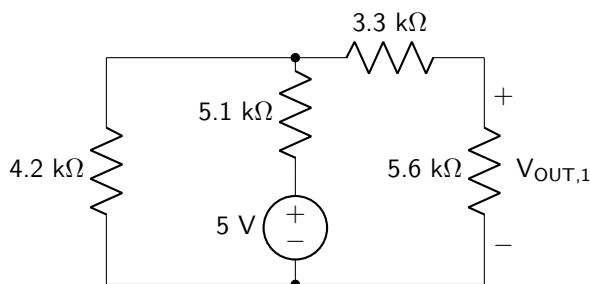
Use superposition to calculate V_{OUT} .



Deactivate the current source (replace it with an open circuit) and calculate the contribution to V_{OUT} from the voltage source.

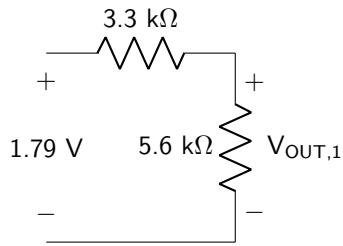


Combine the 1.8 kΩ and 2.4 kΩ resistors in series.



It is necessary to find the voltage drop over the 4.2 kΩ resistor so that a voltage divider can be used to calculate $V_{OUT,1}$. This voltage drop can be calculated by combining the 4.2 kΩ resistor in parallel with the series combination of the 3.3 kΩ and 5.6 kΩ resistors and using a voltage divider.

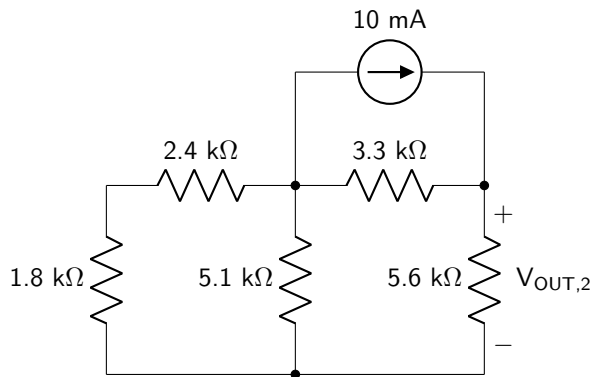
$$\begin{aligned}
 V_{4.2k} &= 5 \text{ V} \left(\frac{4.2 \text{ k}\Omega // 8.9 \text{ k}\Omega}{5.1 \text{ k}\Omega + 4.2 \text{ k}\Omega // 8.9 \text{ k}\Omega} \right) \\
 &= 5 \text{ V} \left(\frac{2.85 \text{ k}\Omega}{5.1 \text{ k}\Omega + 2.85 \text{ k}\Omega} \right) \\
 &= 1.79 \text{ V}
 \end{aligned}$$



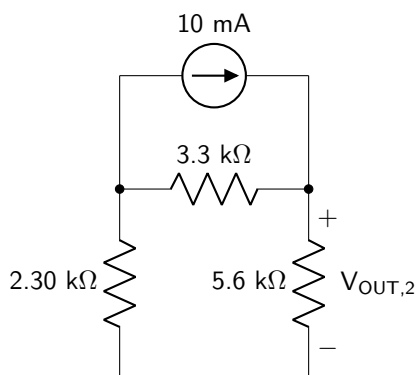
Use the voltage divider rule to calculate $V_{OUT,1}$.

$$V_{OUT,1} = 1.79 \text{ V} \left(\frac{5.6 \text{ k}\Omega}{3.3 \text{ k}\Omega + 5.6 \text{ k}\Omega} \right) = 1.13 \text{ V}$$

Deactivate the voltage source (replace with a short) to determine the contribution to the output voltage due to the current source.



The $1.8 \text{ k}\Omega$ and $2.4 \text{ k}\Omega$ resistors are in series. That combination is in parallel with the $5.1 \text{ k}\Omega$ resistor.



Use a current divider to determine the current flowing through the 5.6 k Ω resistor.

$$\begin{aligned} I_{5.6k} &= 10 \text{ mA} \left(\frac{(5.6 \text{ k}\Omega + 2.30 \text{ k}\Omega) / 3.3 \text{ k}\Omega}{5.6 \text{ k}\Omega + 2.30 \text{ k}\Omega} \right) \\ &= 10 \text{ mA} \left(\frac{2.33 \text{ k}\Omega}{5.6 \text{ k}\Omega + 2.30 \text{ k}\Omega} \right) \\ &= 2.95 \text{ mA} \end{aligned}$$

Use Ohm's law to calculate $V_{OUT,2}$.

$$V_{OUT,2} = (2.95 \text{ mA})(5.6 \text{ k}\Omega) = 16.50 \text{ V}$$

Add the two voltages together to calculate V_{OUT} .

$$V_{OUT} = 1.13 \text{ V} + 16.50 \text{ V} = 17.63 \text{ V}$$

3.2 Source Transformation

Source transformation is a process where a current source can be transformed into a voltage source and vice versa. The transformation process changes the source type and the resistor location, but results in an equivalent circuit. The goal of source transformation is usually to reduce a circuit by changing the position of the source and a resistor; this can lead to further reductions using series and parallel combinations of resistors.

First, it is important to establish what is meant by equivalent circuits. Two equivalent circuits, as shown in figure 3.4, will have identical voltage drops over the load (depicted as V) as well as identical currents flowing into the load (depicted as I).

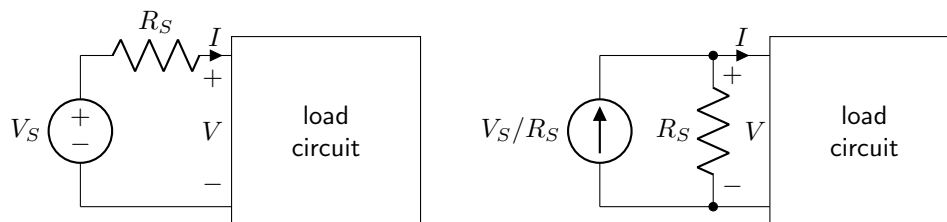


Figure 3.4: Both of these circuits are equivalent as they have identical voltage drops over the load (V) and current flow into the load (I).

A voltage source in series with a resistor can be transformed into an equivalent circuit with a current source in parallel with a resistor. Ohm's law is used to determine the value of the current source, which is equal to V_S/R_S . This is depicted schematically in figure 3.5.

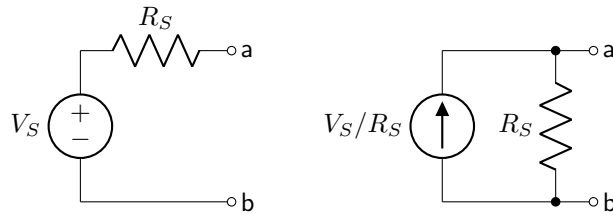


Figure 3.5: The transformation of a voltage source in series with a resistor (left) to a current source in parallel with a resistor (right).

A current source in parallel with a resistor can be transformed into an equivalent circuit with a voltage source in series with a resistor. The voltage source value will be equal to $I_S R_S$ (using Ohm's law). This is depicted schematically in figure 3.6.

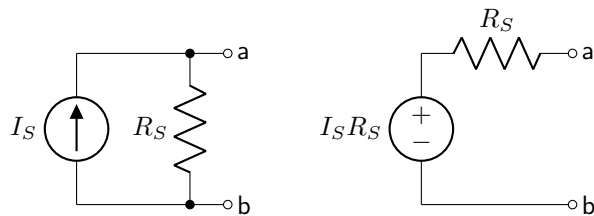


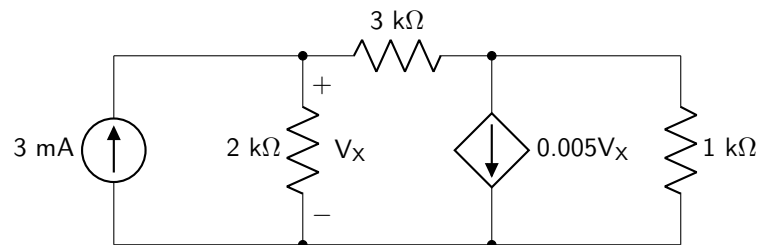
Figure 3.6: The transformation of a current source in parallel with a resistor (left) to a voltage source in series with a resistor (right).

In either case (converting a voltage source to a series source, or vice versa), the position of the resistor changes but the value of the resistor remains unchanged.

It is possible to conduct source transformation with any kind of power source used in a linear circuit. That is: source transformation can be achieved with dependent sources or AC sources. This process is not limited to independent or DC sources.

Example: Source transformation

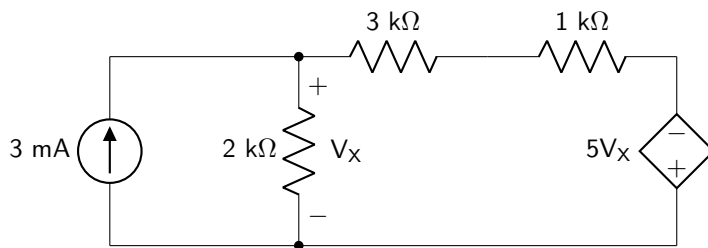
Calculate V_X using source transformation. The units of the dependent source are in A/V.



It is not useful to transform the independent source because the voltage of interest is measured across the 2 kΩ resistor. Instead, the dependent source can be transformed from a VCCS to a VCVS. The new

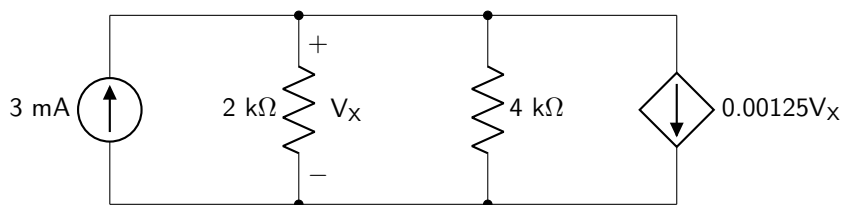
proportionality constant is calculated below.

$$k = (0.005 \text{ A/V})(1000 \text{ } \Omega) = 5$$

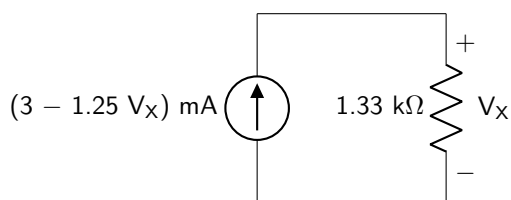


The 3 kΩ and 1 kΩ resistors can be combined in series. Then, the VCVS can be source-transformed back into a VCCS. The new proportionality constant is calculated below.

$$g_m = \frac{5}{4000 \text{ } \Omega} = 0.00125 \text{ A/V}$$



The two resistors can be combined in parallel. Because the two current sources are in parallel, they can be combined (paying close attention to units).



Use Ohm's law to calculate V_X . Units have been removed from the calculations for clarity; all quantities are measured in either mA, V, or kΩ.

$$\begin{aligned} V_X &= (3 - 1.25V_X)(1.3) \\ &= 4 - 1.67V_X \\ &= \frac{4}{2.67} \\ &= 1.5 \text{ V} \end{aligned}$$

3.3 Open-Circuit Voltages and Short-Circuit Currents

Before moving on to more circuit theorems, it is important to discuss the concept of open-circuit voltages and short-circuit currents. These two measurement techniques will be used in abundance when discussing both Thévenin's and Norton's theorems so it is important to have a solid understanding of them.

3.3.1 Open-Circuit Voltage

Open-circuit voltage (known as V_{OC}) is the potential difference between two terminals when they are disconnected from any load circuit. This is shown for a generic circuit in figure 3.7, if the open-circuit voltage is to be measured between the terminals at nodes a and b.

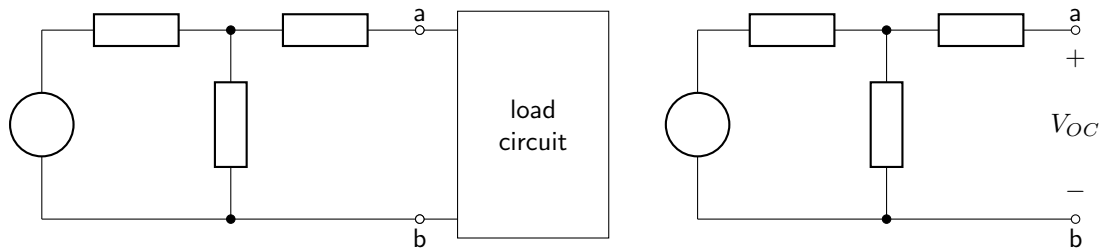
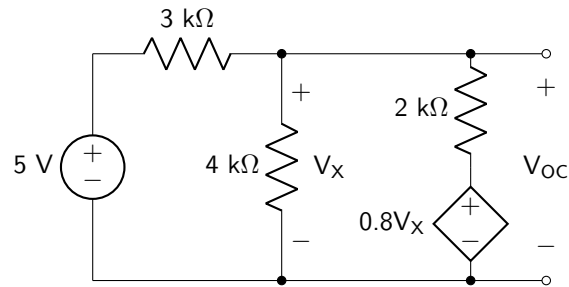


Figure 3.7: To calculate the open circuit voltage (V_{OC}) of the circuit on the left, disconnect the two terminals from the load, as shown on the right.

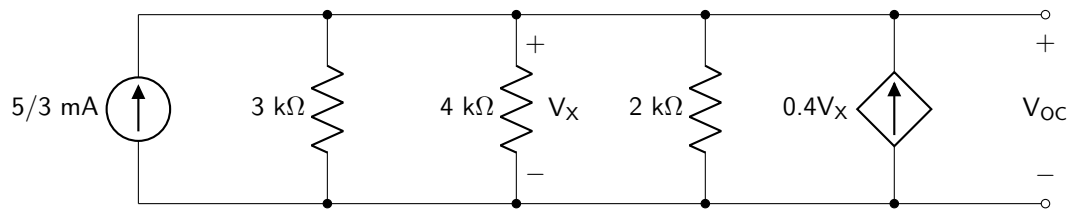
Once the circuit has been re-drawn to disconnect the load, use any circuit law or theorem (Ohm's law, voltage or current divider, mesh analysis, KCL/KVL, superposition, source transformation, etc.) to calculate the open-circuit voltage.

Example: Calculating open-circuit voltage

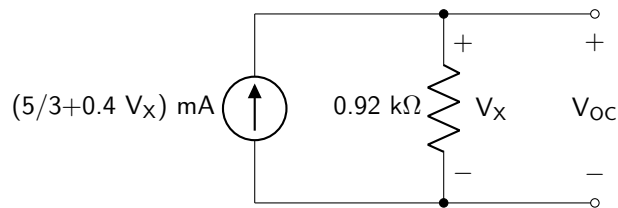
Calculate V_{OC} .



Source transformation will be used to solve this circuit. (Note that KCL/KVL or mesh analysis could also be used.) Both voltage sources will be transformed into current sources. The proportionality constant of the VCCS below is in units of mA/V.



All three resistors can be combined in parallel. Both of the current sources can be added together.



Ohm's law can be used to calculate V_{OC} , which is equal to V_X . All quantities are measured in units of V, mA, or $k\Omega$.

$$\begin{aligned} V_{OC} = V_X &= (5/3 + 0.4V_X)(0.92) \\ &= 1.53 + 0.37V_X \\ &= \frac{1.53}{0.63} \\ &= 2.44 \text{ V} \end{aligned}$$

3.3.2 Short-Circuit Current

Short-circuit current (known as I_{SC}) is the current between two terminals when they are shorted together. This is shown for a generic circuit in figure 3.8, if the short-circuit current is to be measured between the terminals at nodes a and b.

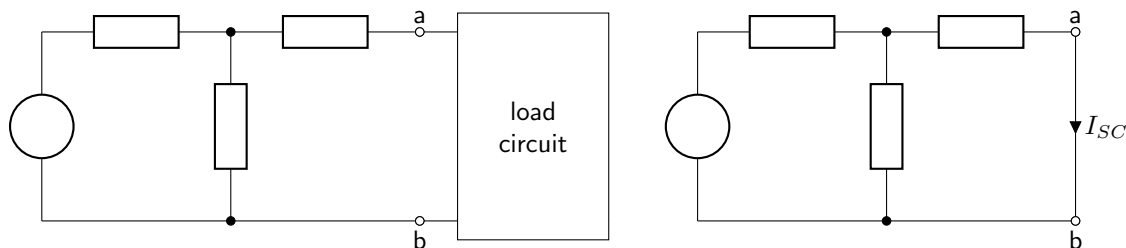


Figure 3.8: To calculate the short circuit current (I_{SC}) of the circuit on the left, short the two terminals together, as shown on the right.

Once the circuit has been re-drawn to short out the load, any circuit law or theorem can be used to calculate the short-circuit current.

At this point, it is important to discuss what happens to shorted-out circuit elements, as this becomes a

source of confusion for students.

If a passive circuit element such as a resistor, capacitor, or inductor is shorted out, the potential difference over that circuit element becomes 0 V. Because the voltage drop is zero, Ohm's law dictates that the current will also be zero. If no current flows through a circuit element, and it contains no voltage drop, then that element is essentially absent from the circuit. Therefore, a shorted passive element can be removed from a circuit diagram for the sake of simplicity. This is depicted in figure 3.9.

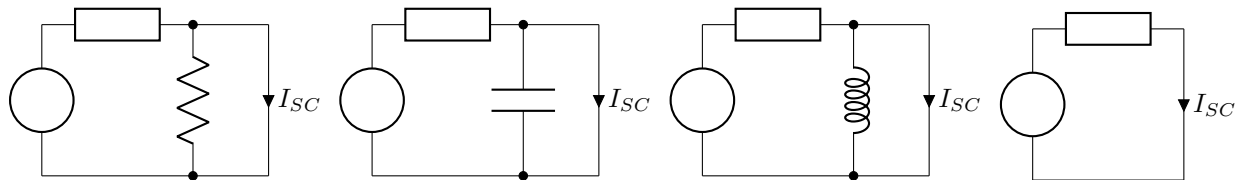


Figure 3.9: Each of these circuits is functionally equivalent.

If a current source is shorted out, the current source continues to contribute current to the circuit. Figure 3.10 shows an example of a shorted current source. Kirchhoff's current law states that the short-circuit current is equal to the sum of the current coming from the current source added to the current supplied by the branch to the left. The current source is providing a useful function and cannot be removed from the circuit. This is true for all kinds of current sources: AC and DC, independent and dependent.

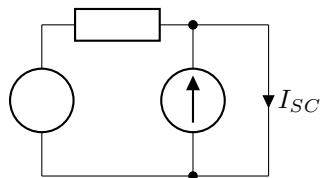


Figure 3.10: A shorted current source cannot be removed from a circuit.

If a voltage source is shorted out (depicted in figure 3.11), it is also not accurate to say that the voltage source is not contributing anything to the circuit. A shorted voltage source will in fact have a very large impact on a circuit. Any voltage source that is directly connected to a low- (or no-) resistance conductor will generate massive (or approaching infinite) amounts of current. This would cause the conductor (and possibly the source itself) to heat up, melt, or start on fire. While it is not a good idea to short a voltage source, it is possible, and it would not be correct to say that the voltage source acts as if it no longer exists.

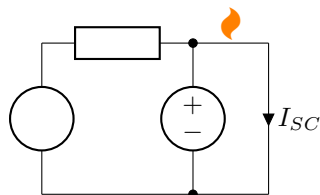
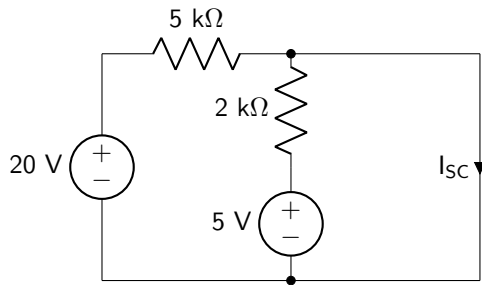


Figure 3.11: While this would be a very bad idea, the voltage source is still contributing much to the circuit, and cannot be removed.

A voltage source in series with a resistor can be source transformed to a current source in parallel with a resistor. Therefore, if a voltage source in series with a resistor is shorted to calculate a short-circuit current, the source-transformed version is equivalent to a combination of passive element shorted out (the series resistor becomes a parallel resistor, shorted to calculate I_{SC}) and a current source shorted out.

Example: Calculating short-circuit current

Calculate I_{SC} .



Mesh analysis will be used to solve this circuit. (It is also possible to use source transformation or KCL/KVL). The mesh on the left will contain clockwise mesh current I_1 . Units have been removed from the calculations; all values are in mA, V, or $k\Omega$.

$$-20 + 5I_1 + 2(I_1 - I_2) + 5 = 0$$

The mesh on the right will contain clockwise mesh current I_2 (which is equal to the short-circuit current).

$$-5 + 2(I_2 - I_1) = 0$$

The matrix, in terms of I_1 , I_2 , and the constant, is given.

$$\begin{bmatrix} 7 & -2 & 15 \\ -2 & 2 & 5 \end{bmatrix} \quad (3.4)$$

The solution to this matrix gives $I_{SC} = I_2 = 6.5$ mA.

3.4 Thévenin's Theorem

Thévenin's theorem provides a very useful way to simplify otherwise complicated circuits. It states that any linear circuit (regardless of the complexity) can be represented by an equivalent circuit that contains a voltage source in series with a resistor. This is depicted in figure 3.12.

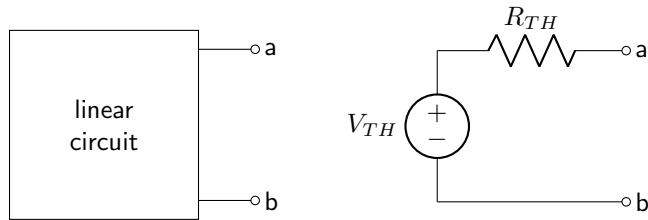


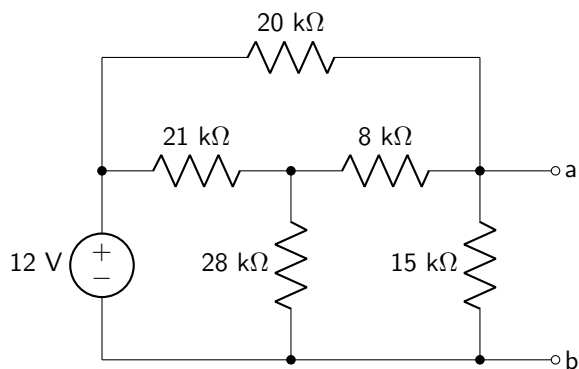
Figure 3.12: Thévenin's theorem states that these two circuits are equivalent, given the correct value of V_{TH} and R_{TH} .

The technique to Thévenin's theorem, then, is to calculate the correct values of the *Thévenin equivalent voltage* (V_{TH}) and *Thévenin equivalent resistance* (R_{TH}). The Thévenin equivalent voltage is simply equal to the open-circuit voltage between the output terminals (labeled a and b in figure 3.12). The procedure to calculate the Thévenin equivalent resistance is given below.

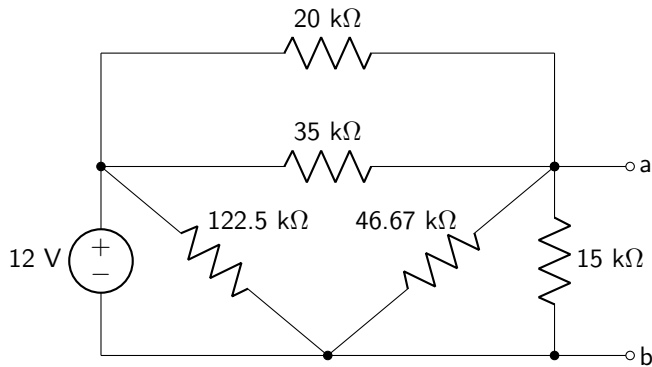
- If there are no dependent sources in the circuit, then R_{TH} is equal to the equivalent resistance seen between the output terminals, with all of the independent sources deactivated.
- If there are dependent sources in the circuit, calculate the short-circuit current (I_{SC}) between the output terminals. The Thévenin equivalent resistance is equal to V_{TH}/I_{SC} .

Example: Thévenin equivalent circuit

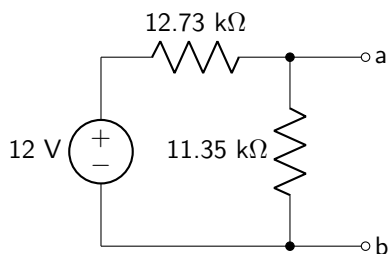
Derive the Thévenin equivalent circuit between nodes a and b.



Perform a wye-delta transform.



Because the $122.5\text{ k}\Omega$ resistor is in parallel with the voltage source, its contribution to the circuit can be ignored. The $20\text{ k}\Omega$ and $35\text{ k}\Omega$ resistors can be combined in parallel. The $46.67\text{ k}\Omega$ and $15\text{ k}\Omega$ resistors can also be combined in parallel. The circuit can be re-drawn.

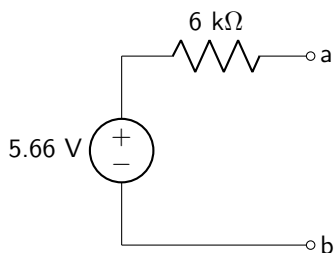


Use a voltage divider to calculate the Thévenin equivalent voltage.

$$V_{TH} = 12\text{ V} \left(\frac{11.35\text{ k}\Omega}{12.73\text{ k}\Omega + 11.35\text{ k}\Omega} \right) = 5.66\text{ V}$$

To calculate the Thévenin equivalent resistance, deactivate the voltage source and calculate the equivalent resistance seen between terminals a and b. That will be equal to both resistors combined in parallel.

$$R_{TH} = 12.73\text{ k}\Omega // 11.35\text{ k}\Omega = 6\text{ k}\Omega$$



3.5 Norton's Theorem

Norton's theorem states that any linear circuit, regardless of the level of complexity, can be represented by an equivalent circuit that contains a current source in parallel with a resistor. This is depicted in figure 3.13.

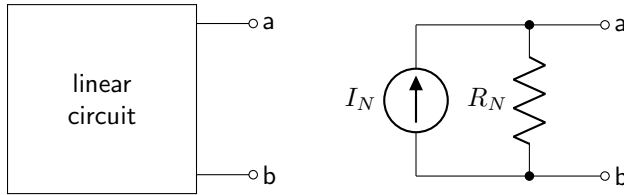


Figure 3.13: Norton's theorem states that these two circuits are equivalent, given the correct value of I_N and R_N .

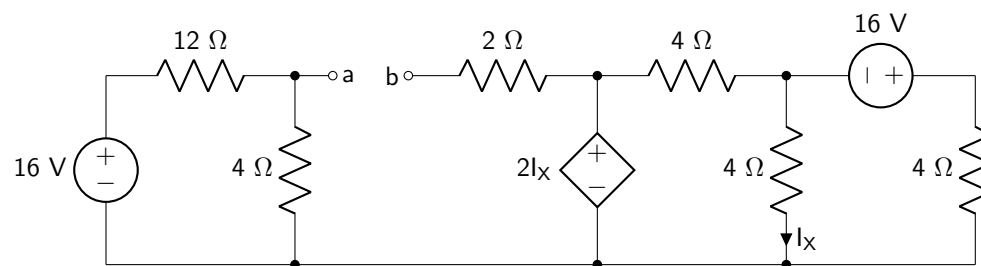
The technique to Norton's theorem is in calculating the *Norton equivalent current* (I_N) and *Norton equivalent resistance* (R_N). The Norton equivalent current is equal to the short-circuit current between the output terminals (labeled a and b in figure 3.13). The procedure to calculate the Norton equivalent resistance is given below.

- If there are no dependent sources in the circuit, then R_N is equal to the equivalent resistance seen between the output terminals, with all of the independent sources deactivated.
- If there are dependent sources in the circuit, calculate the open-circuit voltage (V_{OC}) between the output terminals. The Norton equivalent resistance is equal to V_{OC}/I_N .

It can be noted that Norton's theorem and Thévenin's theorem are identical to each other; one is simply the source-transformed version of the other. It is therefore possible to find a Thévenin equivalent circuit and do a source transformation to derive the Norton equivalent circuit.

Example: Norton equivalent circuit

Derive the Norton equivalent circuit between nodes a and b.



Mesh analysis will be used to solve this circuit. Calculate the open-circuit voltage between nodes a and b. This is equal to $V_a - V_b$. A voltage divider can be used to solve for V_a .

$$V_a = 16 \text{ V} \left(\frac{4 \Omega}{12 \Omega + 4 \Omega} \right) = 4 \text{ V}$$

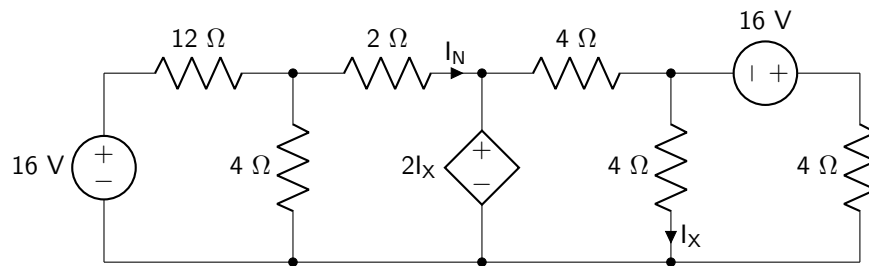
The right-side of this circuit is independent from the left side. Therefore, the two meshes on the right can be used to calculate V_b . I_1 is the mesh current defined clockwise through the CCVS and two $4\ \Omega$ resistors. I_2 is the mesh current defined clockwise through two $4\ \Omega$ resistors and the $16\ \text{V}$ source. The third equation relates the controlling current to the two mesh currents. The matrix, given in form $\alpha I_1 + \beta I_2 + \gamma I_x = \delta$ is shown.

$$\begin{bmatrix} 8 & -4 & -2 & 0 \\ -4 & 8 & 0 & 16 \\ -1 & 1 & 1 & 0 \end{bmatrix} \quad (3.5)$$

When solved, this matrix indicates that $I_x = -1.6\ \text{A}$. V_b is $2I_x = -3.2\ \text{V}$. Now V_{OC} can be calculated.

$$V_{OC} = 4\ \text{V} + 3.2\ \text{V} = 7.2\ \text{V}$$

Calculate the Norton equivalent current.

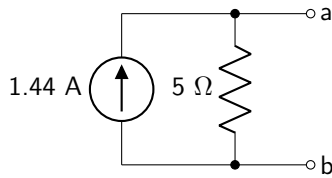


Mesh analysis will be used again. This time, all four meshes will be included in the analysis. The matrix, given in form $\alpha I_1 + \beta I_2 + \gamma I_3 + \delta I_4 + \epsilon I_x = \zeta$ is shown.

$$\begin{bmatrix} 16 & -4 & 0 & 0 & 0 & 16 \\ -4 & 6 & 0 & 0 & 2 & 0 \\ 0 & 0 & 8 & -4 & -2 & 0 \\ 0 & 0 & -4 & 8 & 0 & 16 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{bmatrix} \quad (3.6)$$

The Norton equivalent current is equal to mesh current I_2 . It is therefore equal to $1.44\ \text{A}$. Finally, the Norton equivalent resistance can be calculated.

$$R_N = \frac{7.2\ \text{V}}{1.44\ \text{A}} = 5\ \Omega$$



3.6 Maximum Power Transfer

The theorem of maximum power transfer states that the maximum amount of power will be delivered to the load when the resistance of the load is equal to the Thévenin equivalent resistance of the circuit. The circuit shown in figure 3.14 will be used to demonstrate this theorem.

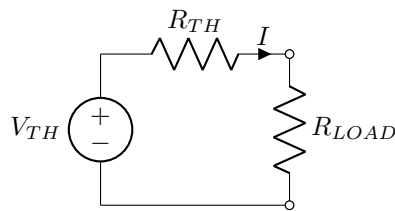


Figure 3.14: Circuit diagram used to demonstrate the theorem of maximum power transfer.

To determine the resistance of the load (R_{LOAD}) that will lead to maximum power transfer, the power consumed by the load can be calculated using the power equation $P = IV$. This is shown in equation 3.7 and derived below.

$$P_{LOAD} = \frac{R_{LOAD}V_{TH}^2}{(R_{TH} + R_{LOAD})^2} \quad (3.7)$$

DERIVATION

$$\begin{aligned} I_{LOAD} &= I = \frac{V_{TH}}{R_{TH} + R_{LOAD}} \\ V_{LOAD} &= V_{TH} \left(\frac{R_{LOAD}}{R_{TH} + R_{LOAD}} \right) \\ P_{LOAD} &= I_{LOAD}V_{LOAD} \\ &= \frac{R_{LOAD}V_{TH}^2}{(R_{TH} + R_{LOAD})^2} \end{aligned}$$

The derivative of equation 3.7 can then be taken with respect to R_{LOAD} . This equation can be set equal to zero to find the maximum of the equation. (The full derivation will not be shown in this book.) The maximum occurs when R_{LOAD} is equal to R_{TH} .

The maximum power that can be consumed by the load can then be calculated, shown in equation 3.8

and derived below.

$$P_{MAX} = \frac{V_{TH}^2}{4R_{TH}} \quad (3.8)$$

DERIVATION

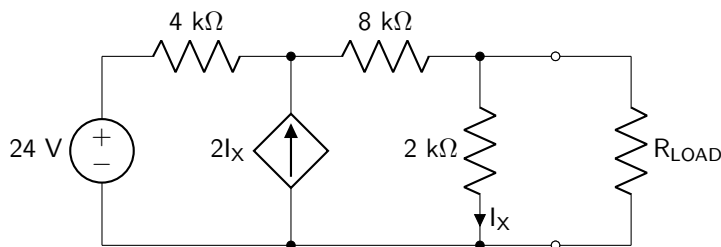
$$\begin{aligned} P_{MAX} &= \frac{R_{TH} V_{TH}^2}{(R_{TH} + R_{TH})^2} \\ &= \frac{R_{TH} V_{TH}^2}{(2R_{TH})^2} \\ &= \frac{R_{TH} V_{TH}^2}{4R_{TH}^2} \\ &= \frac{V_{TH}^2}{4R_{TH}} \end{aligned}$$

Because a Norton equivalent circuit and a Thévenin equivalent circuit are simply source-transformed versions of each other, the maximum power transferred to the load in a Norton equivalent circuit would occur when the load resistance is equal to the Norton resistance.

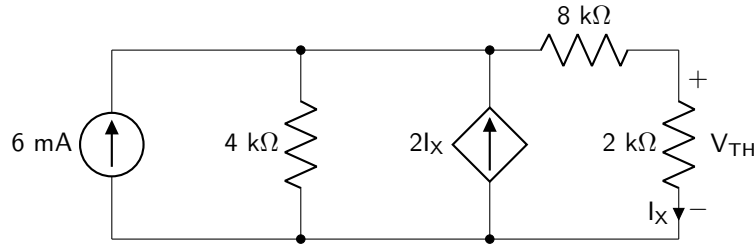
It is important to note that while the maximum *amount* of power is transferred when the load resistance is equal to the Thévenin resistance, this does not equate to the maximum load power *efficiency* (ratio of power delivered to the load to the total power). In the case of maximum power transfer, the power efficiency is limited to 50%.

Example: Maximum power transfer

Calculate the value of the load resistor that is required for maximum power transfer. Then, calculate the maximum power transferred to that load.



Convert the circuit to a Thévenin equivalent circuit. Use source transformation on the 24 V source.



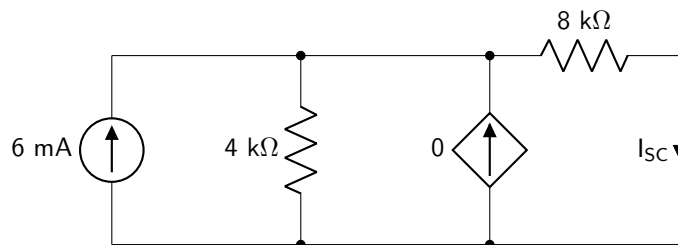
Use the current divider rule to calculate I_X . All units have been removed and are in terms of mA, V, and $k\Omega$.

$$\begin{aligned}
 I_X &= (6 + 2I_X) \left(\frac{4/10}{10} \right) \\
 &= (6 + 2I_X) \left(\frac{4}{14} \right) \\
 &= 1.71 + 0.57I_X \\
 &= \frac{1.71}{0.43} \\
 &= 4 \text{ mA}
 \end{aligned}$$

Use Ohm's law to calculate V_{TH} .

$$V_{TH} = (4 \text{ mA})(2 \text{ k}\Omega) = 8 \text{ V}$$

Calculate the short circuit current. Because the resistor with the controlling current is shorted, the controlling current is 0 mA (no current will flow through the 2 kΩ resistor when it can travel through a zero-resistance short instead), and the dependent source is effectively deactivated.



Use a current divider to calculate the short-circuit current.

$$I_{SC} = 6 \text{ mA} \left(\frac{8 \text{ k}\Omega / 4 \text{ k}\Omega}{8 \text{ k}\Omega} \right) = 2 \text{ mA}$$

Use Ohm's law to calculate the Thévenin equivalent resistance.

$$R_{TH} = \frac{8 \text{ V}}{2 \text{ mA}} = 4 \text{ k}\Omega$$

Therefore, the resistor for maximum power transfer is 4 k Ω . Use equation 3.8 to calculate the maximum power transferred to this load.

$$P_{MAX} = \frac{(8 \text{ V})^2}{4(4 \text{ k}\Omega)} = 4 \text{ mW}$$

Example Problems

Superposition

1. Use superposition to calculate V_X in the circuit shown in figure 3.15.

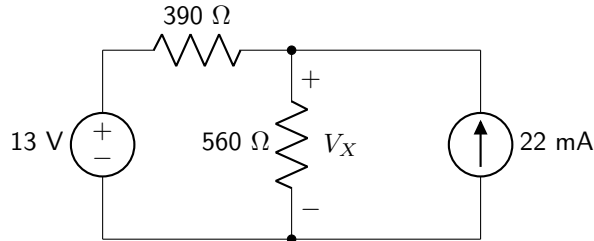


Figure 3.15: Circuit diagram for superposition question 1.

2. Use superposition to calculate V_X in the circuit shown in figure 3.16.

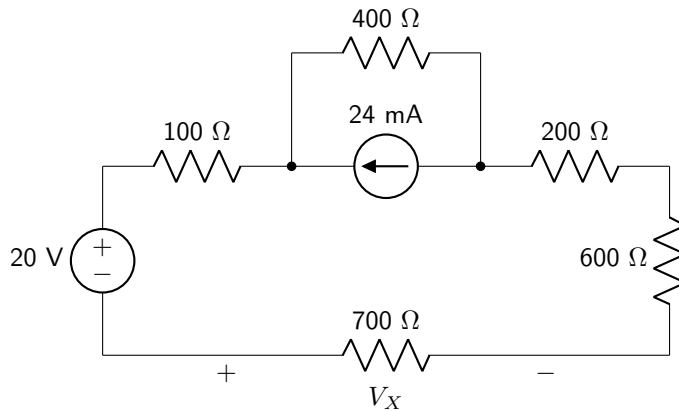


Figure 3.16: Circuit diagram for superposition question 2.

3. Use superposition to calculate I_X in the circuit shown in figure 3.17.

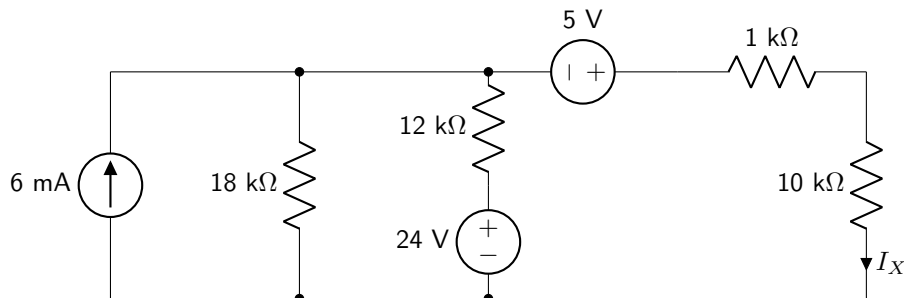


Figure 3.17: Circuit diagram for superposition question 3.

4. Use superposition to calculate V_X in the circuit shown in figure 3.18.

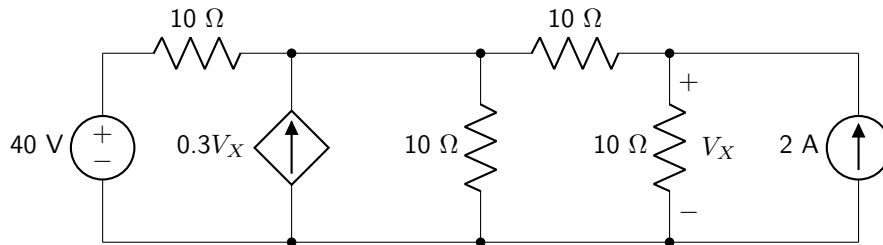


Figure 3.18: Circuit diagram for superposition question 4.

5. Use superposition to calculate V_X in the circuit shown in figure 3.19.

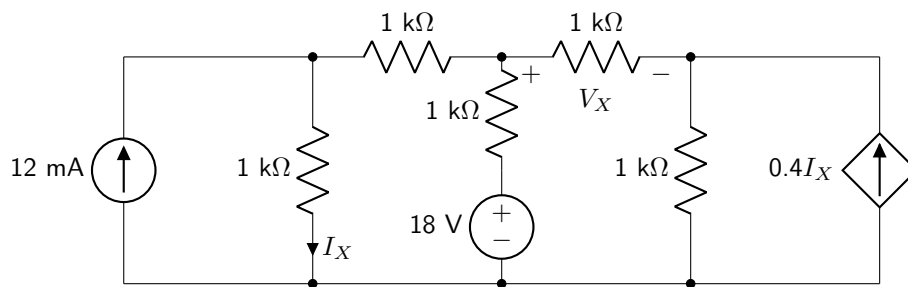


Figure 3.19: Circuit diagram for superposition question 5.

Source Transformation

6. Use source transformation to calculate V_X in the circuit shown in figure 3.15 (in the superposition section).
7. Use source transformation to calculate V_X in the circuit shown in figure 3.16 (in the superposition section).
8. Use source transformation to calculate I_X in the circuit shown in figure 3.17 (in the superposition section).
9. Use source transformation to calculate V_X in the circuit shown in figure 3.18 (in the superposition section).

10. Use source transformation to calculate V_X and V_Y in the circuit shown in figure 3.20.

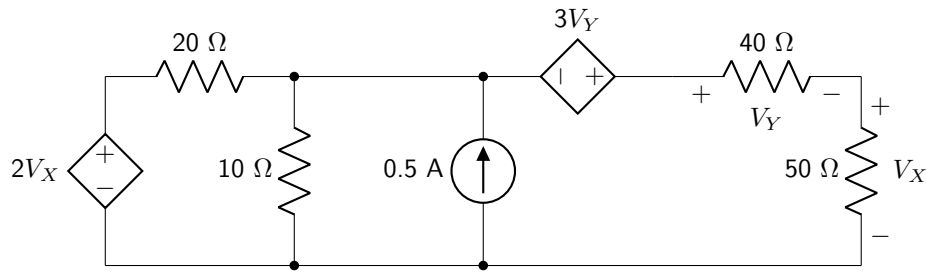


Figure 3.20: Circuit diagram for source transformation question 10.

Thévenin and Norton's Theorems

11. Derive the Thévenin equivalent circuit between nodes a and b in the circuit shown in figure 3.21.

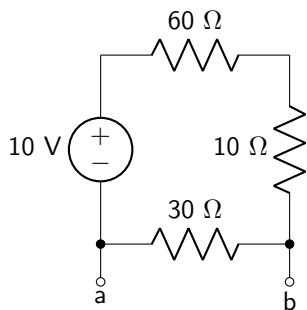


Figure 3.21: Circuit diagram for Thévenin and Norton's theorems question 11.

12. Derive the Norton equivalent circuit between nodes a and b in the circuit shown in figure 3.22.

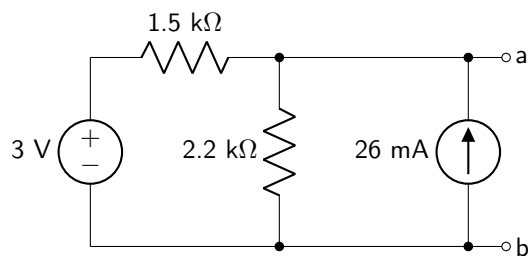


Figure 3.22: Circuit diagram for Thévenin and Norton's theorems question 12.

13. Derive the Thévenin equivalent circuit between nodes a and b in the circuit shown in figure 3.23.

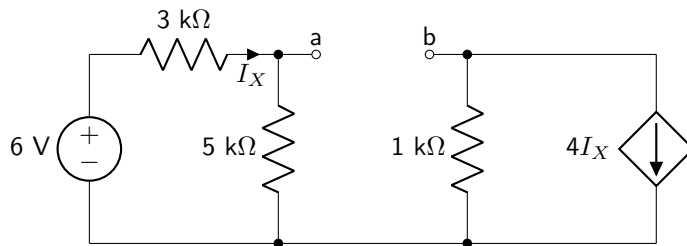


Figure 3.23: Circuit diagram for Thévenin and Norton's theorems question 13.

14. Derive the Thévenin equivalent circuit between nodes a and b in the circuit shown in figure 3.24.

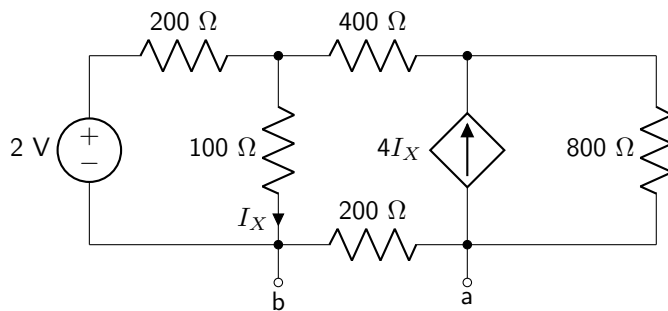


Figure 3.24: Circuit diagram for Thévenin and Norton's theorems question 14.

15. Derive the Norton equivalent circuit between nodes a and b in the circuit shown in figure 3.25.

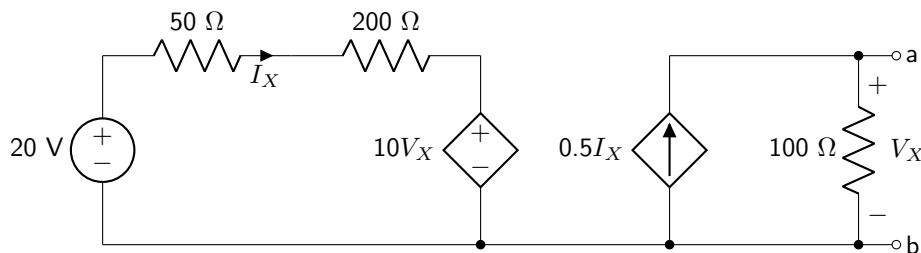


Figure 3.25: Circuit diagram for Thévenin and Norton's theorems question 15.

Maximum Power Transfer

16. Calculate the resistance for maximum power transfer, and the maximum amount of power transferred to the load under that condition, for the circuit shown in figure 3.21 (in the Thévenin and Norton's theorem section).
17. Calculate the resistance for maximum power transfer, and the maximum amount of power transferred to the load under that condition, for the circuit shown in figure 3.22 (in the Thévenin and Norton's theorem section).

18. Calculate the resistance for maximum power transfer, and the maximum amount of power transferred to the load under that condition, for the circuit shown in figure 3.23 (in the Thévenin and Norton's theorem section).
19. Calculate the resistance for maximum power transfer, and the maximum amount of power transferred to the load under that condition, for the circuit shown in figure 3.24 (in the Thévenin and Norton's theorem section).
20. Calculate the resistance for maximum power transfer, and the maximum amount of power transferred to the load under that condition, for the circuit shown in figure 3.25 (in the Thévenin and Norton's theorem section).

4 Operational Amplifiers (Op-Amps)

An operational amplifier (henceforth referred to as an op-amp) is a voltage amplifier that typically uses divided negative feedback to generate a particular gain value. There are a lot of technical details to unpack in that sentence, so this chapter will start with the premise that the reader has no knowledge of amplifiers at all, and move slowly up from the concept of amplification to the functioning of op-amps. In addition, op-amps are the first active circuit element to be discussed in this book. The implications of that quality will be noted in this chapter.

Commercially available op-amps are integrated circuits that contain all of the transistors and resistive components needed to create the device. Op-amps will be treated as a whole monolithic entity; as this textbook does not explore non-linear elements such as transistors, the inner workings of op-amps will not be discussed.

Note: in this chapter of the textbook, voltages will be noted at single nodes. However, voltage is always relative. There is no such thing as a voltage at one point. When discussing voltage at a node in this chapter, it is implied that that voltage is being measured with respect to ground. This will make the circuit diagrams and discussions of voltage much simpler in this chapter.

4.1 Amplifiers

Amplifiers will be discussed in this section. More specifically, this section will discuss single-ended input voltage amplifiers. That means that there is one input signal (and one output signal – all of the amplifiers discussed in this chapter will be single-ended output), and the property of the circuit being amplified is the voltage. The schematic for an amplifier is shown in figure 4.1. Note the presence of connections for a supply voltage (both positive: V^+ , and negative: V^-), a consequence of an amplifier being an active circuit element.

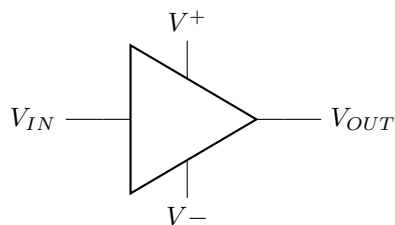


Figure 4.1: Circuit schematic for a single-ended input amplifier.

An amplifier takes an input voltage (V_{IN}) and multiplies it by some value (called the gain) and passes that voltage to the output (V_{OUT}). The mathematical relationship between output and input, with gain equal to A , is shown in equation 4.1.

$$V_{OUT} = AV_{IN} \quad (4.1)$$

This amplification cannot happen indefinitely; the output voltage cannot exceed the supply voltage (in either the positive or negative direction). In other words: $V_{OUT} \not> V^+$ and $V_{OUT} \not< V^-$. This is what is meant by an active circuit element. The amplification does not come out of thin air, it comes from the presence of a power supply connected to the supply terminals.

The input and output voltages are plotted for a single-ended input amplifier with a gain of $A = 10$ in figure 4.2, given a supply voltage of ± 15 V. Note how the voltage output *saturates* at the values of the supply voltage. The output cannot exceed the supply.

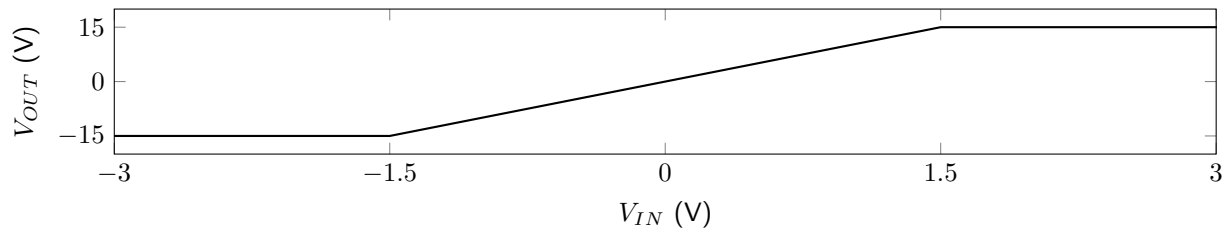


Figure 4.2: Input / output voltage characteristic of a single-ended input amplifier with gain (A) of 10 and supply voltage of ± 15 V.

There is some important terminology to understand that is used to explain the characteristics of an amplifier based on the value of the gain. These characteristics are not limited to single-ended input amplifiers but are relevant to all of the devices discussed in this chapter.

- An amplifier is **non-inverting** if the gain is positive ($A > 0$).
- An amplifier is **inverting** if the gain is negative ($A < 0$).
- An amplifier is **attenuating** if the absolute value of the gain is less than one ($|A| < 1$).
- An amplifier is **amplifying** (or non-attenuating) if the value of the gain is greater than or equal to one ($|A| \geq 1$).
- An amplifier is **saturated** when the output voltage is equal to the supply voltage ($V_{OUT} = V^+$ or $V_{OUT} = V^-$).

4.2 Differential Amplifiers

A differential amplifier has two inputs. The output is scaled to the difference of the two inputs (hence the term differential), multiplied by the gain. This relationship is expressed in equation 4.2. V_N is the voltage at the inverting input of the amplifier, and V_P is the voltage at the non-inverting input of the amplifier.

$$V_{OUT} = A(V_P - V_N) \quad (4.2)$$

The schematic of a differential amplifier is shown in figure 4.3.

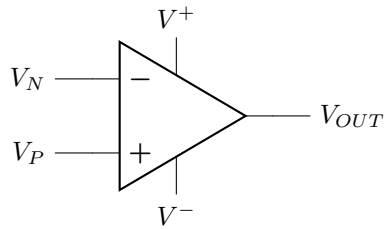


Figure 4.3: Circuit schematic for a differential amplifier.

As the inverting input increases, the output becomes more negative. As the non-inverting input increases, the output becomes more positive. Regardless, the output voltage cannot exceed the supply in either direction.

The circuit model for a differential amplifier, that uses only components that have already been discussed in this book, is shown in figure 4.4. R_{IN} corresponds to the input resistance of the amplifier and R_{OUT} corresponds to the output resistance of the amplifier. The gain of the circuit is depicted as A . The supply is not shown. It should be emphasized that this circuit diagram corresponds to a model of how a differential amplifier works, and does not contain the actual circuit elements (generally, transistors and resistors) that are used to create one.

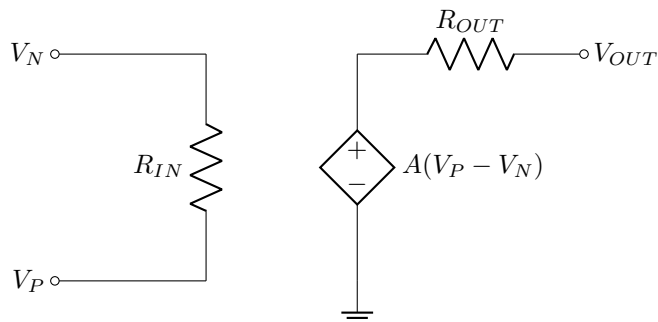


Figure 4.4: Circuit model of a differential amplifier.

An operational amplifier (op-amp) is a differential amplifier with three key features: large gain, large input resistance, and small output resistance. The large gain means that the differential input is multiplied by a very large number before being passed to the output. This concept will be discussed in section 4.3. A large input resistance implies that very little (ideally no) current flows in to either the inverting or non-inverting inputs into the amplifier's internal circuitry. A low output resistance implies that the voltage drop over and current through any load circuit will be minimally affected by the op-amp itself.

4.3 Comparator Circuits

A comparator is an op-amp that is built to exploit the high gain and subsequent quick saturation of output voltage upon any difference between the inverting and non-inverting inputs. To demonstrate this quality,

consider a differential amplifier (shown in figure 4.5) with a gain of 200,000. (This gain is typical for commercial comparators such as the LM339.) The supply voltage is limited to ± 15 V.

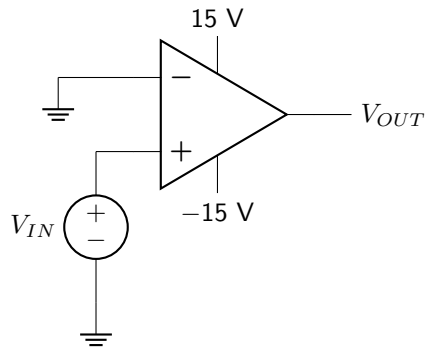


Figure 4.5: Circuit schematic for a comparator.

With a gain of 200,000, the output voltage will saturate very quickly upon application of an input voltage. In fact, the input voltage V_{IN} that will cause a saturated output will be $15 \text{ V}/20000 = 75 \times 10^{-6} \text{ V}$. If a negative input voltage is applied, saturation will be achieved on the output if a voltage with an amplitude larger than $75 \times 10^{-6} \text{ V}$ is applied to the input. A graph of the input and output voltage characteristics is shown in figure 4.6.

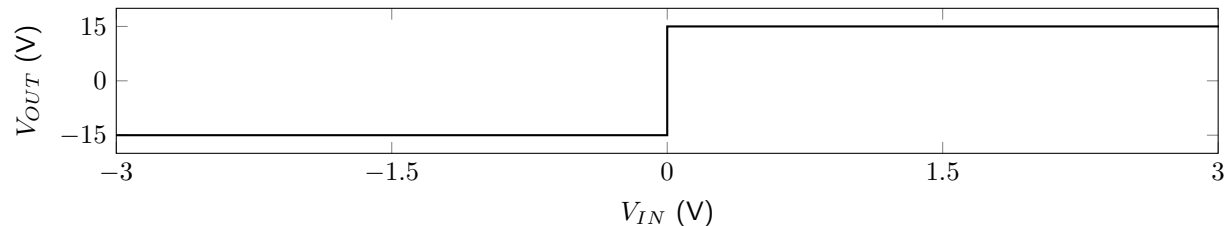


Figure 4.6: Input / output voltage characteristic of a comparator with gain of 200,000 and supply voltage of ± 15 V.

This device saturates any time the voltage drop with respect to ground on the inverting input is greater than the voltage drop with respect to ground on the non-inverting input. What practical application does a comparator have? As the name implies, a comparator is used to determine if the voltage on the non-inverting input is greater than the voltage on the inverting input. By itself, this is a useful circuit element that can be used to judge the magnitude of two electrical signals. Two other important applications are explained in the next two subsections of this book.

Before discussing comparator applications, it is important to note what happens to the functioning of a comparator if all three characteristics of an op-amp are not met. These three characteristics are discussed in the context of a positive saturation value (i.e. $V_{OUT} = V^+$), but the principles still apply for negative saturation ($V_{OUT} = V^-$).

First: a comparator with a small gain will create a non-trivial threshold voltage that must be achieved on the non-inverting input before the output saturates. With a gain of 200,000 (as we assumed in our example

comparator above) the threshold voltage is $75 \mu\text{V}$ (assuming a supply of 15 V), which is, from a practical standpoint, within a rounding error of zero. The relationship between threshold voltage (V_{THR}), positive supply voltage (V^+) and gain (A) is given in equation 4.3. It is clear to see that as the gain decreases, the threshold voltage increases. The larger the threshold voltage, the less the comparator is able to compare the two input voltage magnitudes, especially when the magnitude difference is small.

$$V_{THR} = \frac{V^+}{A} \quad (4.3)$$

Second: a comparator with a large output resistance will limit the current that flows through the output, and therefore will reduce the maximum voltage drop of the output to less than the value of the positive supply voltage. The maximum output voltage ($V_{OUT,MAX}$) that can be achieved over a load resistor (R_{LOAD}) given a non-zero output resistance (R_{OUT}) is shown in equation 4.4. As the output resistance increases, the maximum output voltage decreases. This equation makes the assumption that there is no current leakage from other parts of the comparator (such as from the input terminals), which would complicate matters.

$$V_{OUT,MAX} = V^+ \left(\frac{R_{LOAD}}{R_{OUT} + R_{LOAD}} \right) \quad (4.4)$$

Third: a comparator with a small input resistance will lead to a small amount of current entering into the amplifier internals through the non-inverting and inverting inputs. This current will lead to an error voltage between the two input voltages V_P and V_N . This is not a huge issue for comparators, but will become an important factor in divided feedback op-amp circuits.

Comparators vs. “Op-Amps”

While a comparator is an operational amplifier, they are generally used in circuits that require specific design parameters that are not adequately addressed by using a generic “op-amp” chip. When purchasing an integrated circuit (IC) to use in a design, it is important to purchase the correct chip that is capable of carrying out the required functions. When building a circuit that requires a comparator, it can be advantageous to seek out a specific IC known as a “comparator” rather than just purchasing a generic IC known as an “op-amp,” even though, as discussed, comparators are technically a subset of the op-amp category.

First, comparators are meant to be operated in the saturation region, whereas op-amps are meant to be operated in the linear region between V^- and V^+ . Therefore, it is important for comparators to have a quick saturation response. Comparator chips are generally designed to have much faster switching characteristics than generic op-amps.

Second, noise can become a problem in comparator circuits. If a small amount of noise is present on the input, it can cause the output to swing between both saturation modes. In order to prevent that, many comparator chips are designed to incorporate hysteresis on the input signal, which requires

a threshold of change to be met on the input before the output will swing between voltages.

4.3.1 Analog to Digital Conversion (ADC)

An analog to digital converter (ADC) takes an analog voltage and turns it into a signal suitable for use in digital electronics. Generally, digital electronics use a constant voltage of 5 V (sometimes 3.3 V) to represent a logical HIGH signal, and 0 V to represent a logical LOW signal. The binary number system and Boolean algebra dictate the functioning of digital systems. (For more information, refer to my [Digital Systems textbook](#).)

Without getting into the proverbial weeds, it is sufficient to note that an ADC will translate an analog voltage (capable of representing any value between 0 V and 5 V) into a digital signal (one or more values that are either HIGH or LOW). There are many types of ADCs that can be designed, most of which contain one or more comparators. For simplicities sake, only the *flash-type* ADC will be explored in this textbook.

A flash-type ADC is capable of directly converting an analog voltage into a digital signal (other types of ADCs may require a finite sampling time followed by some time period to complete the conversion before the next sample can be taken). It also uses a relatively simple to understand architecture. Of course, these benefits must indicate a major drawback. Indeed, a flash-type ADC requires $2^n - 1$ comparators to create an n -bit ADC (which is capable of generating binary numbers between 0 and $2^n - 1$).

The schematic of a 3-bit flash-type ADC is given in figure 4.7. Note that the individual supply connections are not shown for any of the comparators. The positive supplies are connected to V^+ while the negative supplies are connected to ground.

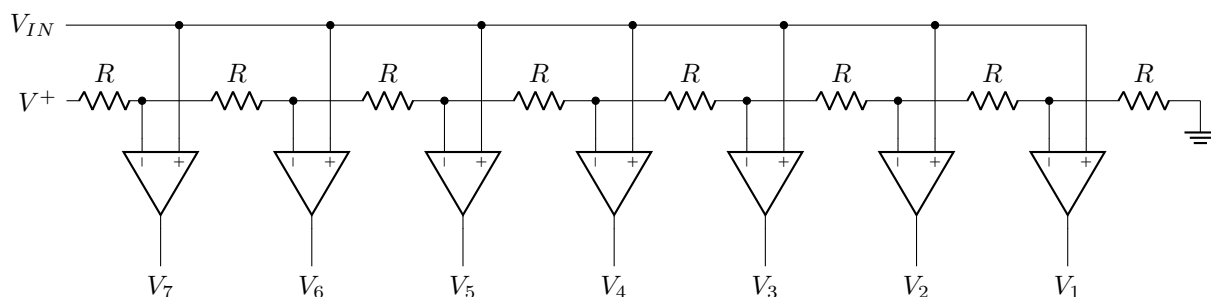
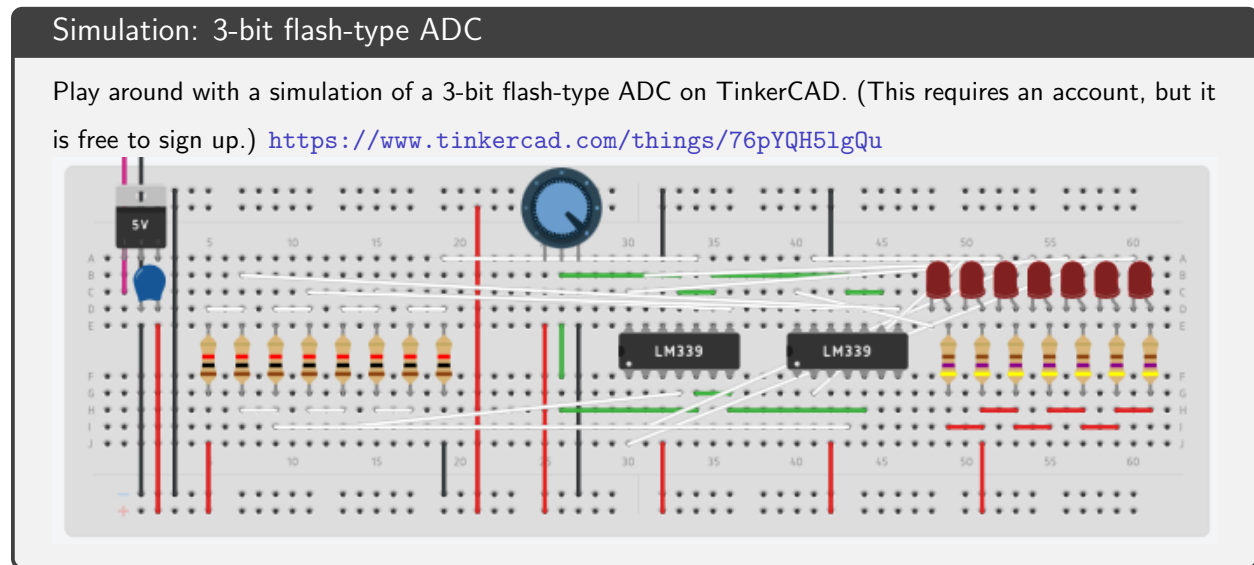


Figure 4.7: A 3-bit flash-type analog to digital converter (ADC).

The input voltage (V_{IN}) can be any analog voltage between 0 V and V^+ . The source voltage (V^+) will typically be 5 V and is connected to one end of a large resistor ladder. The resistor ladder divides the source voltage into 2^n (in this example, eight) steps. (Therefore, each of the resistors needs to have an identical value, denoted on the circuit diagram as R .) As the input voltage exceeds each one of these voltage steps, the corresponding comparator output will saturate at V^+ (logic HIGH). While the output does not correspond to a binary number (which would require a digital logic device called an encoder), it does result in a digital

number (values that only use logic LOW and logic HIGH voltage levels).



4.3.2 Pulse-Width Modulation (PWM)

Pulse-width modulation (PWM) is a means of changing the average voltage of a signal without changing the minimum or maximum voltage levels that are used. In this manner, a digital electronic system (which, as mentioned, only uses 0 V and 5 V signals) can vary the amount of time the signal is HIGH with respect to the amount of time the signal is LOW to modify the average voltage of the signal. PWM isn't just used in digital electronics. It is also an effective way of decreasing the average power supplied to a load while delivering a large maximum power by sending short pulses instead of a continuous signal. (This method is used in laser optics when a continuous laser beam would burn a hole in a sample, but repeated short pulses will not.) In addition, there are devices that specifically require a PWM signal to function properly (such as some servomotors).

PWM signals have multiple properties. First is the frequency of the pulses, which corresponds to the number of pulses sent per unit of time. Second is the duty cycle, which corresponds to the fraction of time the signal is HIGH compared to the period of the waveform. The equation for duty cycle (D) is shown in equation 4.5. T_{HIGH} is the amount of time the waveform is held HIGH every cycle, and T_{LOW} is the amount of time the waveform is held LOW every cycle. (The period of the wave is equal to $T_{HIGH} + T_{LOW}$ and is equal to the inverse of the frequency.)

$$D = \frac{T_{HIGH}}{T_{HIGH} + T_{LOW}} \quad (4.5)$$

The average voltage \bar{V} of a PWM signal is given in equation 4.6, where D is the duty cycle, V_{MAX} is the

maximum value of the modulated signal, and V_{MIN} is the minimum value of the modulated signal.

$$\bar{V} = DV_{MAX} + (1 - D)V_{MIN} \quad (4.6)$$

While PWM is accomplished using a completely different mechanism in digital electronics such as microcontrollers, it is possible to create a PWM circuit using a comparator, as shown in figure 4.8. The input voltage (V_{IN}) can be a sinusoid or triangle wave. The reference voltage (V_{REF}) can be tuned to change the duty-cycle of the output voltage (V_{OUT}). The output will saturate between V^- and V^+ .

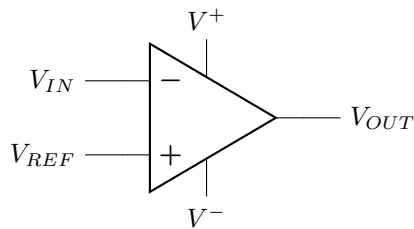


Figure 4.8: Circuit schematic of a comparator used to generate a PWM signal.

Example PWM waveforms given three different reference voltages with the same sinusoidal input signal are shown in figure 4.9. As the reference voltage is increased, the duty cycle of the PWM waveform increases.

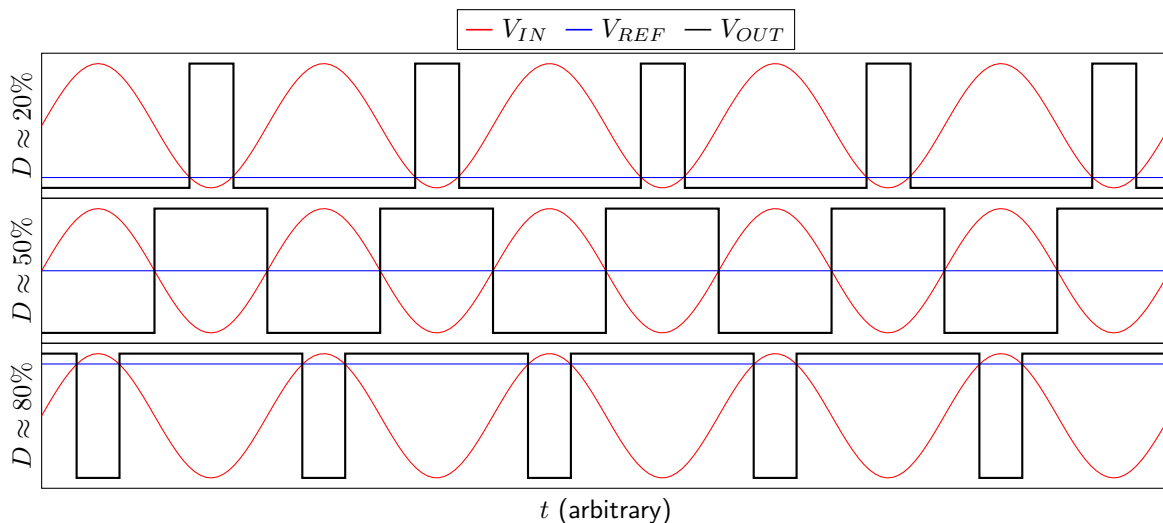
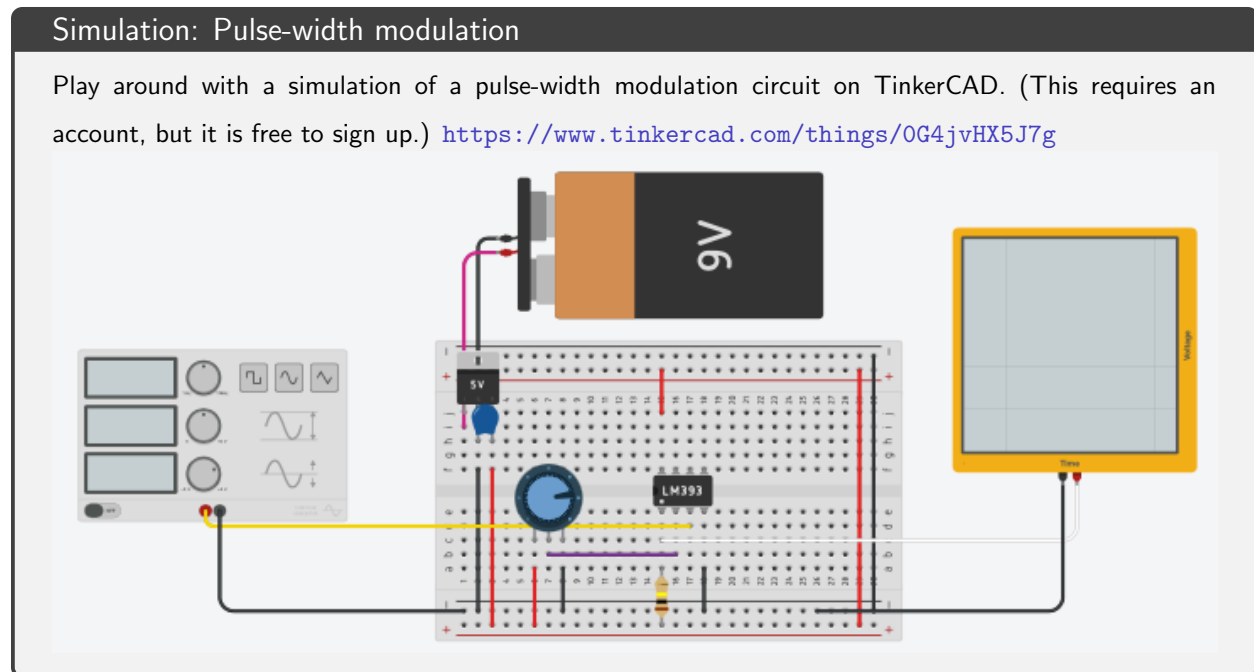


Figure 4.9: Three PWM signals (black waveform) with varying duty cycles. The input wave V_{IN} is represented by a red curve and the reference voltage V_{REF} is represented by a blue line.

In this example, the frequency of the PWM waveform is set by the frequency of V_{IN} . This makes a flexible PWM circuit where the minimum and maximum voltage values of the output are set by the comparator supply pins (V^+ and V^-), the frequency is set by the input waveform, and the duty cycle is set by the reference voltage.



4.4 Feedback Circuits

Feedback is a mechanism used to regulate, or control, the output of an electronic device. There are two types of feedback: positive and negative.

Positive feedback is what happens when a speaker and microphone come in close proximity to each other. As sound goes into the microphone, it emits from the speaker. When the speaker is close to the microphone, the microphone will take that sound emission from the speaker and amplify it, which gets picked up by the microphone, and so on, until a loud squeal is emitted when the maximum amplification setting (saturation) is reached. Without the physical limitations to this process, energy could theoretically be amplified infinitely. This is undesirable.

Negative feedback is a process that regulates itself. The difference between a setpoint and a measured value is used to change the output value. For example, if a car's cruise control is set at 50 MPH, and the car is only traveling at 30 MPH, there is a difference of 20 MPH. This difference then controls how much to engage the throttle. As the car's speed increases, the difference between the measured speed and setpoint decreases, causing the throttle to engage proportionally less. At some point, the setpoint will be achieved, the difference will be 0 MPH, and no more throttle adjustments need to be made. Because it is self-regulating, negative feedback forms the basis for modern control systems.

Negative feedback is also used in op-amp circuits. Because the gain of an op-amp is so high, it is impractical on its own as an amplifier, due to the fact that it saturates so rapidly (which was a feature in comparator circuits, but a drawback otherwise). However, feedback from the output to the inverting input can be used to tame the gain to a value that has practical uses.

Note that the remaining circuit diagrams in this chapter will not necessarily explicitly show the voltage supply connections. In a real circuit, supply connections must exist to power the op-amps. They are merely hidden in these diagrams to make them easier to read.

4.4.1 Voltage Follower

A voltage follower circuit is the simplest type of negative feedback op-amp circuit. The output is fed into the inverting input, creating the relationship given in equation 4.7.

$$V_{OUT} = A(V_P - V_{OUT}) \quad (4.7)$$

The schematic of a voltage follower is shown in figure 4.10.

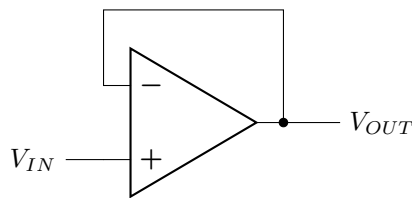


Figure 4.10: Schematic of a voltage follower circuit.

Assume that an input of 5 V is applied to V_{IN} while V_{OUT} is initially 0 V. The op-amp has a gain of 200,000. Plugging this information into equation 4.7, the voltage difference of 5 V multiplied by 200,000 means that the output will attempt to increase dramatically. Before the output can reach a value of one million, or even the positive supply voltage, however, it will increase slightly. (The rate at which the voltage levels increases is known as the *slew rate*.) Let's say the output increases to 1 V. Now the voltage difference is only 4 V, and the output will try to drive to $4 \text{ V} \times 200,000$. Before that happens, the voltage will increase a bit more, say, to 2 V. The voltage change now is only 3 V, so the output will not drive as high. At a certain point, the output will reach a stable level as close as possible to the input voltage of 5 V.

This circuit is called a voltage follower because the stability of the negative feedback system causes the output voltage to follow the input voltage. The *effective gain* of this circuit is one; the output equals the input without any multiplicative factor. What is the purpose of a circuit that does not change the value of the input voltage before passing it to the output? The purpose of a voltage follower is that the output follows the input while being effectively isolated from it. To understand what this means, consider the following example.

Let's say that a 9 V battery needs to be used to create a stable output voltage of 5 V, over which a resistive load will be placed (but the exact resistance of the load is unknown and might be variable). One way that a circuit designer could reduce a 9 V source to 5 V would be to create a voltage divider. This voltage divider is shown in figure 4.11.

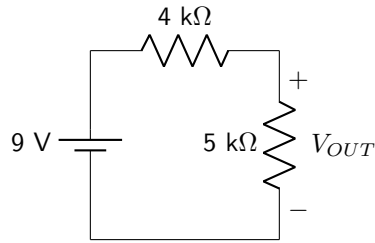


Figure 4.11: Voltage divider to convert a 9 V input into a 5 V output.

Once a load is connected to the output of the voltage divider, however, the equivalent resistance of the voltage divider circuit changes. The output voltage will never be equal to the desired 5 V value, unless the resistance of the load is infinite! The output voltage relationship, for this specific circuit, is given in equation 4.8. Because the load resistance can be variable, it is not possible to engineer the voltage divider to account for the load resistance.

$$V_{OUT} = 9 \text{ V} \left(\frac{5R_{LOAD}}{9R_{LOAD} + 20} \right) \quad (4.8)$$

There needs to be another way to solve this problem. Fortunately, the voltage follower is the solution. Because of the negative feedback driving the output to be equal to the input, a load resistor will not affect the output voltage in any way. The circuit diagram for this solution is shown in figure 4.12.

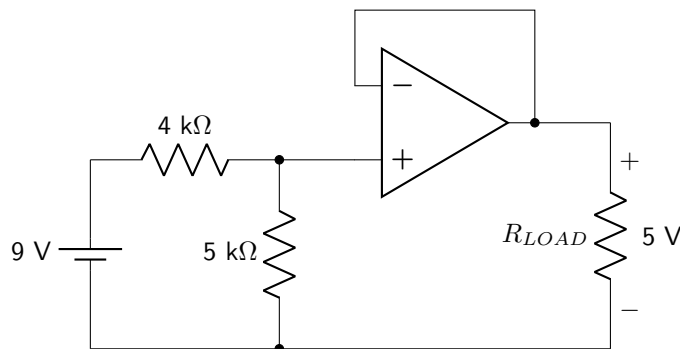


Figure 4.12: A voltage divider followed by a voltage follower will convert a 9 V input into a 5 V output regardless of the load resistance.

When connected as shown in figure 4.12, there will be a 5 V drop over the load resistor. Ohm's law states that current will flow through the resistor. That current does not come from the battery, as the input resistance of an op-amp is very high (ideally infinite). Where does the current come from? It comes from the internal circuitry of the op-amp, which draws from the supply voltage. Remember: the op-amp is an active circuit element. (Refer to section 4.6 for more information on op-amp output current. The load resistor will be ultimately limited by the amount of current the op-amp can source.)

4.5 Divided Feedback Op-Amp Circuits

The voltage follower has a gain of one. An op-amp without negative feedback has an extremely large gain. To exploit the shades of gray in between one and infinity, *divided feedback* can be used. A schematic of a generic divided feedback circuit is shown in figure 4.13.

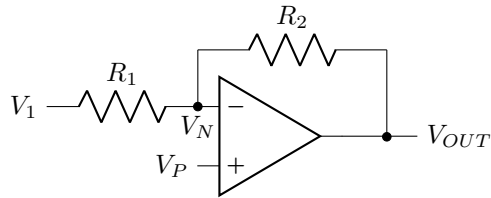


Figure 4.13: A divided feedback circuit.

Although the negative feedback path includes a resistor now (denoted as R_2 in figure 4.13), the same property that forced the output voltage to be equal to the input voltage forces the voltage difference between the inverting and non-inverting inputs to be as close to zero as possible. The voltage at V_N will therefore be driven to be (ideally) exactly equal to V_P . This is called a *virtual node*; the two voltages are identical without being physically connected.

There are two very important properties of divided feedback op-amp circuits that come from the ideal op-amp characteristics (infinite gain, infinite input resistance, zero output resistance): **the voltage at both the inverting and non-inverting nodes is equal; and no current enters into either the inverting or non-inverting inputs of the op-amp.** These two properties will enable us to analyze all of the following op-amp circuits in this textbook.

4.5.1 Inverting Op-Amp

An inverting op-amp modifies the input voltage with the equation $V_{OUT} = AV_{IN}$, where the gain A is a negative number. A schematic of an inverting op-amp is shown in figure 4.14.

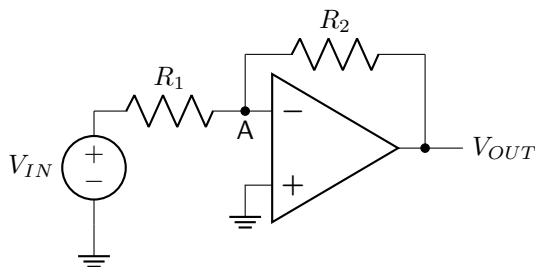


Figure 4.14: Circuit diagram of an inverting op-amp.

The basic design of an inverting op-amp requires two resistors to create divided feedback. The non-inverting input is connected to ground. Using the two properties of an ideal op-amp, the gain (A) of this op-amp can be calculated.

First, KCL at node A (depicted on the circuit diagram: node A is the point at which both resistors connect to the inverting input of the op-amp) tells us that the current flowing through resistor R_1 must be equal to the current flowing through resistor R_2 . This is true because no current enters into the op-amp. Second, the voltages at each node in the circuit can be determined. At node A, the voltage is 0 V (because of the virtual node of the circuit; another of the ideal op-amp properties). We can now use Ohm's law to determine the relationship between V_{OUT} and V_{IN} , shown in equation 4.9 and derived below.

$$A = -\frac{R_2}{R_1} \quad (4.9)$$

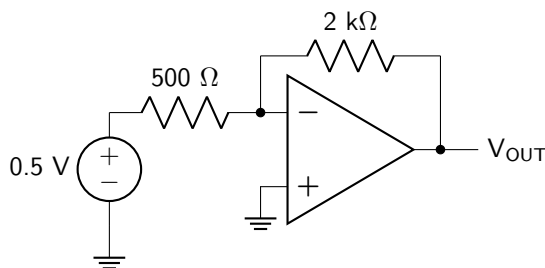
DERIVATION

$$\begin{aligned} I_{R_1} &= I_{R_2} \\ \frac{V_{IN} - 0}{R_1} &= \frac{0 - V_{OUT}}{R_2} \\ V_{OUT} &= -\frac{R_2}{R_1} V_{IN} \\ A &= -\frac{R_2}{R_1} \end{aligned}$$

The gain of an inverting op-amp is therefore equal to $-R_2/R_1$. Note that it is possible for an inverting op-amp to attenuate or amplify a signal. However, the output signal will always be inverted.

Example: Inverting op-amp

Calculate the gain and output voltage of the op-amp circuit. Assume that the supply pins are connected to a sufficiently high voltage not to saturate the output.



This circuit is an inverting amplifier. The gain is calculated below.

$$A = \frac{-2000 \Omega}{500 \Omega} = -4$$

Multiply the input voltage by the gain to determine the output voltage.

$$V_{OUT} = (-4)(0.5 \text{ V}) = -2 \text{ V}$$

4.5.2 Non-Inverting Op-Amp

A non-inverting op-amp also modifies the input voltage with the equation $V_{OUT} = AV_{IN}$, except this time we will expect the value of A to be a positive value. The schematic for a non-inverting op-amp circuit is shown in figure 4.15.

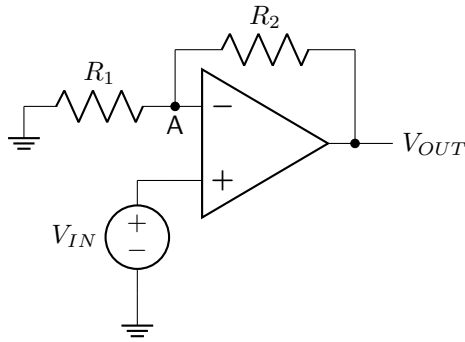


Figure 4.15: Circuit diagram of a non-inverting op-amp.

The same two ideal op-amp properties will be used to determine the gain of this circuit. KCL at node A indicates that the current through resistor R_1 is equal to the current through resistor R_2 , because no current enters into the inverting input into the op-amp itself. The voltage at node A is equal to V_{IN} due to the virtual node between inverting and non-inverting inputs. Ohm's law can be used to determine the circuit properties, shown in equation 4.10 and derived below.

$$A = 1 + \frac{R_2}{R_1} \quad (4.10)$$

DERIVATION

$$\begin{aligned} I_{R_1} &= I_{R_2} \\ \frac{0 - V_{IN}}{R_1} &= \frac{V_{IN} - V_{OUT}}{R_2} \\ V_{OUT} &= \left(1 + \frac{R_2}{R_1}\right) V_{IN} \\ A &= 1 + \frac{R_2}{R_1} \end{aligned}$$

The gain of a non-inverting op-amp is therefore equal to $1 + R_2/R_1$. Note that it is not possible for an inverting op-amp to attenuate a signal, only amplify it.

4.5.3 Cascaded Op-Amps

It is possible to cascade op-amps; the output of one op-amp is fed into the input of a second op-amp. Each op-amp is called a stage. The op-amps can be analyzed independently and multiplied to determine the

overall circuit gain, or the entire circuit can be analyzed. Consider two cascaded inverting op-amps, shown in figure 4.16.

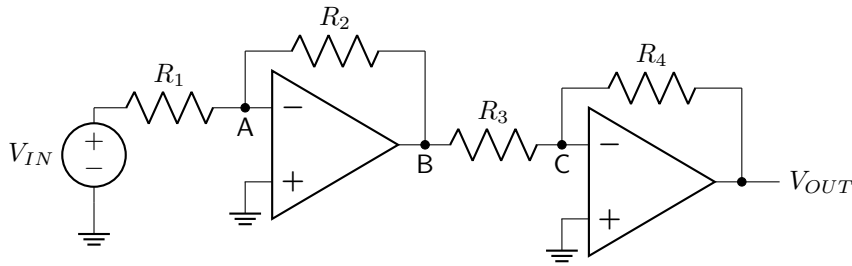


Figure 4.16: Two cascaded inverting op-amps.

It is possible to analyze this circuit by using the ideal op-amp properties. The current through R_1 is equal to the current through R_2 . The current through R_3 is equal to the current through R_4 . (The current through R_2 is **not** necessarily equal to the current through R_3 , however!) Additionally, the voltage at nodes A and C are equal to 0 V due to the virtual nodes with the non-inverting inputs. The circuit output voltage is shown in equation 4.11 and derived below.

$$V_{OUT} = \left(-\frac{R_4}{R_3}\right) \left(-\frac{R_2}{R_1}\right) V_{IN} \quad (4.11)$$

DERIVATION

$$\begin{aligned} I_{R_1} &= I_{R_2} \\ \frac{V_{IN} - 0}{R_1} &= \frac{0 - V_B}{R_2} \\ V_B &= -\frac{R_2}{R_1} V_{IN} \\ I_{R_3} &= I_{R_4} \\ \frac{V_B - 0}{R_3} &= \frac{0 - V_{OUT}}{R_4} \\ V_{OUT} &= \left(-\frac{R_4}{R_3}\right) \left(-\frac{R_2}{R_1}\right) V_{IN} \end{aligned}$$

This confirms that the gain of the cascaded circuit is equal to the product of the gains of the individual stages.

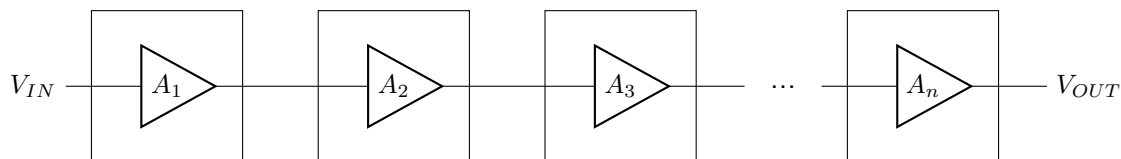


Figure 4.17: Block diagram of n cascaded op-amp circuits.

In general, and assuming that saturation does not occur at any stage, n op-amps cascaded as shown in the block diagram in figure 4.17, with individual gains of A_1 through A_n will have a gain equal to the product of all of the individual gains, as expressed in equation 4.12.

$$A_{TOTAL} = \prod_{i=1}^n A_i \quad (4.12)$$

Cascading op-amps can be beneficial to increase overall circuit gain. Multiple stages can also be useful in active filter circuits (which will be discussed in section 9.3 in this book). Realistically, non-idealities in op-amps may cause compounding issues when op-amps are cascaded together, as non-idealities in the early stages will be amplified by subsequent stages.

4.5.4 Summing and Difference Amplifiers

A summing amplifier has an output proportional to the sum of multiple input circuits. A schematic of a summing op-amp circuit is shown in figure 4.18.

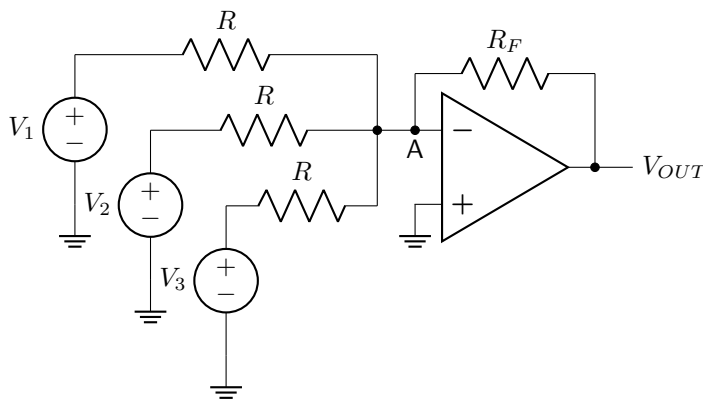


Figure 4.18: Circuit diagram of a summing amplifier.

Using KCL at node A and Ohm's law, it is possible to find the output equation of the op-amp circuit, given in equation 4.13.

$$V_{OUT} = -\frac{R_F}{R}(V_1 + V_2 + V_3) \quad (4.13)$$

Note that the inverting configuration of this op-amp circuit means that the output is proportional to the negative sum of the input voltages. If it's necessary to have a non-inverting configuration, either a non-inverting op-amp circuit can be used, or the inverting summing amplifier can be cascaded into an inverting amplifier, creating a two-stage circuit.

The schematic for a difference amplifier is shown in figure 4.19.

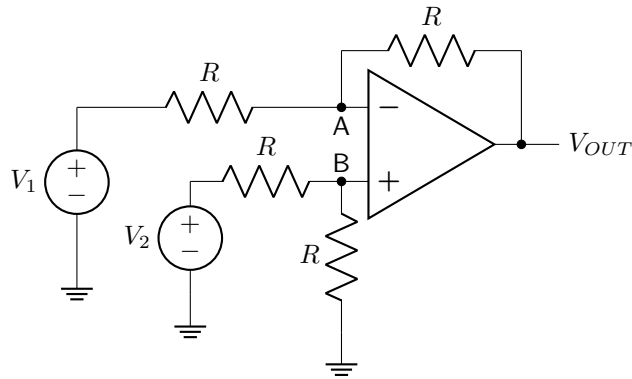


Figure 4.19: Circuit diagram of a difference amplifier.

It is again possible to use KCL and Ohm's law to find the output equation of the circuit. The voltage at node A (V_A in equation 4.14) is equal to the voltage at node B due to the virtual node. The solution is given in equation 4.14 and derived below.

$$V_{OUT} = -(V_1 - V_2) \quad (4.14)$$

DERIVATION

$$\begin{aligned} \frac{V_1 - V_A}{R} &= \frac{V_A - V_{OUT}}{R} \\ \frac{V_2 - V_A}{R} &= \frac{V_A}{R} \\ V_{OUT} &= -(V_1 - V_2) \end{aligned}$$

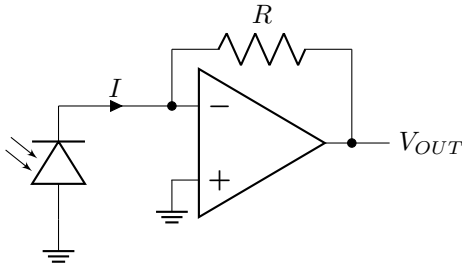
This is an inverting unity gain difference amplifier. To achieve a different gain value, it is possible to use different resistor values, but the circuit would need to be re-analyzed. It is also possible to cascade an inverting unity gain difference amplifier with a second stage to achieve a different gain value.

4.5.5 General Op-Amp Circuits

Not all op-amp circuits fall into the categories discussed above (inverting, non-inverting, summing, etc.). In those cases, careful analysis can lead to a mathematical understanding of the circuit properties. First: it must be emphasized that the op-amp properties (virtual node, no current entering the inputs) are only valid if there is negative feedback present in the circuit. In the absence of negative feedback, then those two assumptions cannot be used to solve for the circuit properties. However, if the presence of negative feedback is established, then circuit analysis tools can be used with the virtual node and zero-current input properties to solve for the circuit characteristics.

Example: Current-sensing circuit

A photodiode is a circuit device that converts light to electric current. It is used as the input to an op-amp circuit, as shown. Calculate the output voltage and its relationship to I , the current generated by the photodiode.



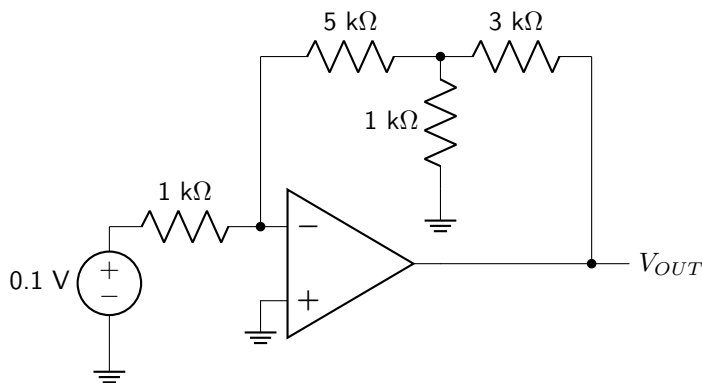
First, it must be established that there is negative feedback included in this circuit. The presence of the resistor between the output voltage and the inverting input means that we can use the assumptions of an op-amp circuit to solve this.

The voltage at the inverting input must be equal to 0 V, as the non-inverting input is connected to ground. As no current can enter into the inverting input of the op-amp, the amount of current flowing through the resistor must be equal to I . Therefore a relationship between the output voltage and photodiode current can be derived.

$$V_{OUT} = -IR$$

Example: T-network circuit

In order to generate a large value of circuit gain in an inverting amplifier, either the feedback resistor must be very large, the input resistor must be very small, or both. A T-network is used to generate a large circuit gain without this constraint on resistor values.



Because there is feedback between the output of the circuit and the inverting input, op-amp assumptions

can be used to solve this circuit.

The voltage at the inverting input must be equal to 0 V. The current through the 1 k Ω input resistor can therefore be calculated.

$$I = \frac{0.1 \text{ V}}{1 \text{ k}\Omega} = 0.1 \text{ mA}$$

All of this current must run through the 5 k Ω resistor (as no current can enter through the inverting input on the op-amp). The voltage at the T junction can be calculated using Ohm's law.

$$V_T = -(0.1 \text{ mA})(5 \text{ k}\Omega) = -0.5 \text{ V}$$

The current flow through the 1 k Ω resistor in the feedback network can now be calculated. Because the voltage at V_T is negative, we know that current must flow "upward" through the resistor, and we will treat that as the positive direction.

$$I_{1k} = \frac{0.5 \text{ V}}{1 \text{ k}\Omega} = 0.5 \text{ mA}$$

Using KCL at the T junction, we can calculate that the current flowing through the 3 k Ω resistor must be the sum of the two currents.

$$I_{3k} = 0.5 \text{ mA} + 0.1 \text{ mA} = 0.6 \text{ mA}$$

Now Ohm's law can be used to calculate the output voltage.

$$I = \frac{V_T - V_{OUT}}{R} = \frac{-0.5 \text{ V} - V_{OUT}}{3 \text{ k}\Omega} = 0.6 \text{ mA}$$

$$V_{OUT} = -[(0.6 \text{ mA})(3 \text{ k}\Omega) + 0.5 \text{ V}] = -2.3 \text{ V}$$

The gain of this circuit is -23.

4.5.6 Implementing Mathematical Functions with Op-Amps

Op-amps can be used to implement many different mathematical functions. Addition and subtraction can be accomplished using summing and difference amplifiers, as discussed in section 4.5.4. Simultaneous sets of addition and subtraction can be solved by using more than one op-amp, as discussed in section 4.5.7.

Multiplication by a constant is accomplished by configuring an inverting (or non-inverting) amplifier circuit to have an amplifying gain. Division by a constant is accomplished by configuring an amplifier circuit to have an attenuating gain. (Note: it is very difficult computationally to multiply two variable values together. This is a constraint not limited to analog computation; it is also difficult in digital computers.)

Differentiation and integration can be implemented by using capacitors in the amplifier circuit, either in the feedback or input path. These two circuits will be discussed in section 5.5.

While non-linear functions are outside the scope of this class, a diode has a natural logarithm response between voltage and current. Therefore a diode in an amplifier circuit can be used to implement either a $\ln(x)$ or e^x response.

4.5.7 Solving Linear Equations

It is possible to use op-amps to solve equations. In fact, op-amps were used in analog computers to perform calculations on many diverse physical quantities including ballistic projectile trajectory, aircraft stability and control, and even economic models (MONIAC computer). While a detailed history and explanation of analog computation is outside of the scope of this textbook, it makes for fascinating reading.

This book will consider two scenarios of equation-solving. First, consider a single linear equation, shown in equation 4.15.

$$ax + b = 0 \tag{4.15}$$

An op-amp circuit can be used to solve this equation for x . Consider the inverting op-amp shown in figure 4.14. In this circuit, the coefficient a is the ratio of the feedback resistor to the input resistor: $a = R_2/R_1$. The constant term b comes from the value of the voltage source: $b = V_{IN}$. Finally, the output voltage contains the value of the variable x .

Simultaneous linear equations can be solved by using more than one op-amp and using the features of a summing amplifier circuit. To solve two simultaneous linear equations, two op-amps are used. The output of each op-amp is fed into both inverting inputs after being scaled by a potentiometer. A third input into each inverting input comes from a voltage source and provides the constant terms. A very simplified circuit diagram is shown in figure 4.20.

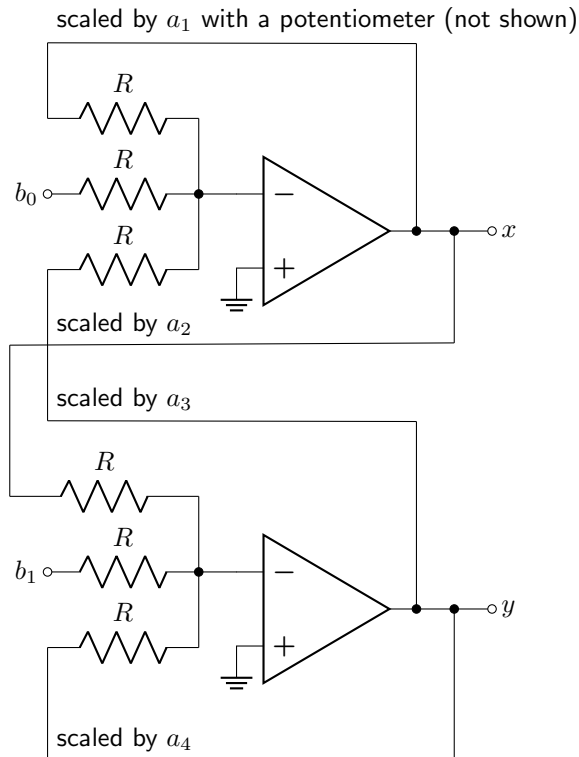


Figure 4.20: Op-amp circuit that can solve two simultaneous linear equations.

Scaling potentiometers (not shown) are used to scale each output voltage to a coefficient before it is passed to a resistor (labeled R in figure 4.20). Voltage supplies are used to generate the constant terms. This circuit can solve the equations shown in equations 4.16–4.17.

$$0 = a_1x + a_2y + b_0 \quad (4.16)$$

$$0 = a_3x + a_4y + b_1 \quad (4.17)$$

In order to generate negative coefficients, it would be necessary to include additional inverting amplifiers to generate values of $-x$ and $-y$ to use on the scaling potentiometers. It also must be noted that there are limitations with this circuit. There is a maximum and minimum value of voltage that can occur on the output, which is limited by the supply voltage. Therefore this equation-solving circuit cannot solve for arbitrarily large numbers. However, it can solve a scaled version of an equation. The output would then be multiplied by that scaling factor to obtain the correct output.

Simulation: Linear equation-solver

Play around with a simulation of a linear equation-solver on CircuitJS: <http://www.falstad.com/circuit/circuitjs.html>. Download the simulation file: <https://drive.google.com/file/d/1bRtaGDRFJcZW7cvv-BPm65j89Ihp1env>. Then open CircuitJS, and click on File ... Open File. Select the downloaded file. You can change the values of each coefficient by moving the sliders on each potentiometer.

It is also possible to solve differential equations using this method. This would require an op-amp building block that can take the integral of a voltage. Non-linear equations can be solved by designing op-amp building blocks that are capable of computing the desired function (say, using a diode to calculate natural logarithms).

4.6 Op-Amp Output Current

Up until this point, the discussion about op-amps has mostly been confined to discussions of voltages and voltage amplification. However, Ohm's law states that where there is voltage (and a closed path), there is current. Where does the current flowing through the load of an op-amp circuit come from? Consider the voltage follower shown in figure 4.21.

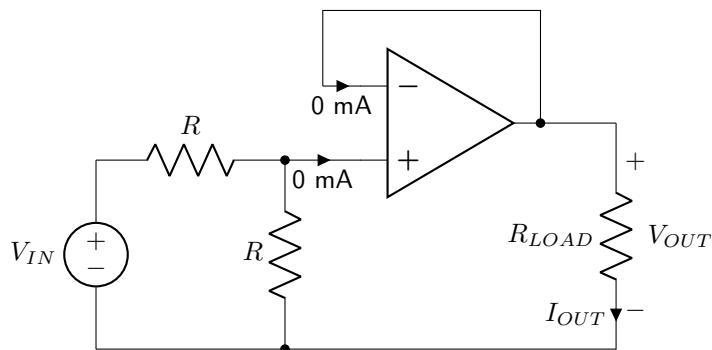


Figure 4.21: A voltage follower circuit with current flow into and out of the op-amp labeled.

In an ideal situation, no current flows into the inverting or non-inverting input terminals of an op-amp. This means that the feedback path of the voltage follower contributes no current to the load. Where does I_{OUT} come from? It must be sourced from the op-amp itself. The current comes from the power supply that's connected to the op-amp supply terminals, and then flows through the output of the op-amp into the load. This property of an integrated circuit to supply current is called *sourcing current*. The op-amp of the voltage follower sources current. The amount of current that an op-amp can source is listed on the device datasheet.

In an inverting amplifier, the output current flows in the opposite direction, as shown in figure 4.22.

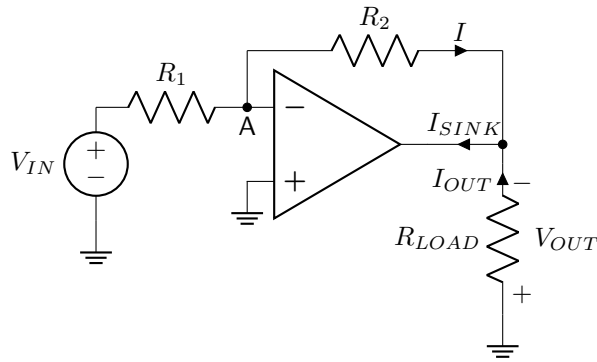


Figure 4.22: Circuit diagram of an inverting op-amp with a resistive load, showing the direction of each current.

No current flows into the op-amp inputs. Now that the input voltage source is not electrically isolated from the output (as it was in the voltage follower circuit), the source is able to generate current. The op-amp will now have to accept current through its output node. This is known as *sinking current*. The amount of current the op-amp must sink can be calculated by using KCL at the output node, shown in equation 4.18 and derived below.

$$I_{SINK} = \frac{V_{IN}}{R_1} \left(1 + \frac{R_2}{R_{LOAD}} \right) \quad (4.18)$$

DERIVATION

$$\begin{aligned} I_{SINK} &= I + I_{OUT} \\ &= \frac{V_{IN}}{R_1} + \frac{R_2 V_{IN}}{R_1 R_{LOAD}} \\ &= \frac{V_{IN}}{R_1} \left(1 + \frac{R_2}{R_{LOAD}} \right) \end{aligned}$$

The amount of current that an op-amp can sink is listed on the datasheet of the device. Figure 4.23 shows an example excerpt of the electrical characteristics section of the datasheet for the LM324 op-amp. It shows the minimum and typical values of output current that can be sourced or sunk under certain test settings. This particular op-amp is capable of sourcing much more current than it can sink. Paying attention to these characteristics is very important if particular current sourcing or sinking requirements must be met.

Electrical Characteristics

PARAMETER		TEST CONDITIONS	LM324		UNIT
			MIN	TYP	
Output Current	Source	$V_{IN}^+ = 1\text{ V}, V_{IN}^- = 0\text{ V},$ $V^+ = 15\text{ V}, V_o = 2\text{ V}, T_A = 25^\circ\text{C}$	20	40	mA
	Sink	$V_{IN}^- = 1\text{ V}, V_{IN}^+ = 0\text{ V},$ $V^+ = 15\text{ V}, V_o = 2\text{ V}, T_A = 25^\circ\text{C}$	10	20	μA
		$V_{IN}^- = 1\text{ V}, V_{IN}^+ = 0\text{ V},$ $V^+ = 15\text{ V}, V_o = 200\text{ mV}, T_A = 25^\circ\text{C}$	12	50	μA

Figure 4.23: Example excerpt of the electrical characteristics section of the LM324 op-amp datasheet.

Some datasheets list the output current specifications under a section labeled “output short-circuit current” rather than “output current” as shown in figure 4.23.

Extra Resources

- Moghimi, Reza. Ask The Applications Engineer–31: Amplifiers as Comparators? April 2003. <https://www.analog.com/en/analog-dialogue/articles/amplifiers-as-comparators.html>
- O'Donnell, Bill. How to Read an Op-Amp Data Sheet. www.physics.unlv.edu/~bill/PHYS483/op_amp_datasheet.pdf
- Texas Instruments. Understanding Basic Analog – Ideal Op Amps. July 1999. Revised October 2016. <https://www.ti.com/lit/an/slaa068b/slaa068b.pdf>

Example Problems

1. Calculate the output voltage and gain of the circuit shown in figure 4.24. (Assume that the supply voltage is sufficient to generate any output value.)

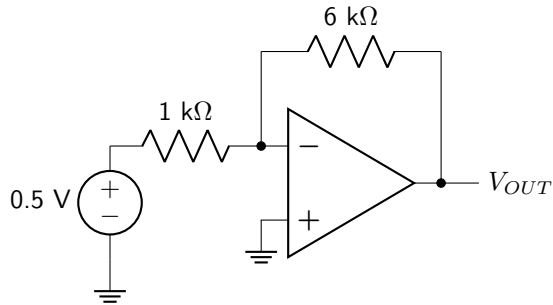


Figure 4.24: Circuit diagram for op-amps question 1.

2. Calculate the output voltage of the circuit shown in figure 4.25. (Assume that the supply voltage is sufficient to generate any output value.)

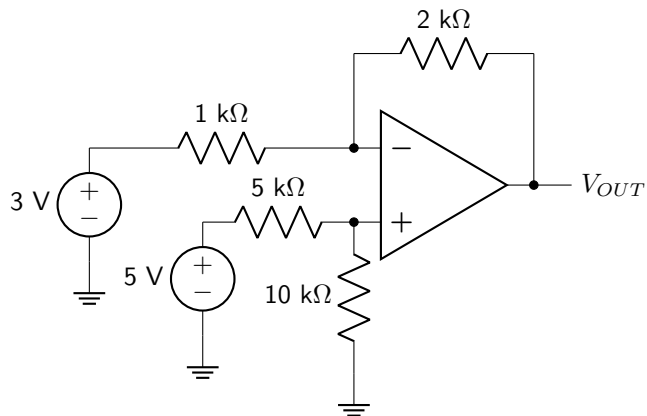


Figure 4.25: Circuit diagram for op-amps question 2.

3. Calculate the output voltage and gain of the circuit shown in figure 4.26. (Assume that the supply voltage is sufficient to generate any output value.)

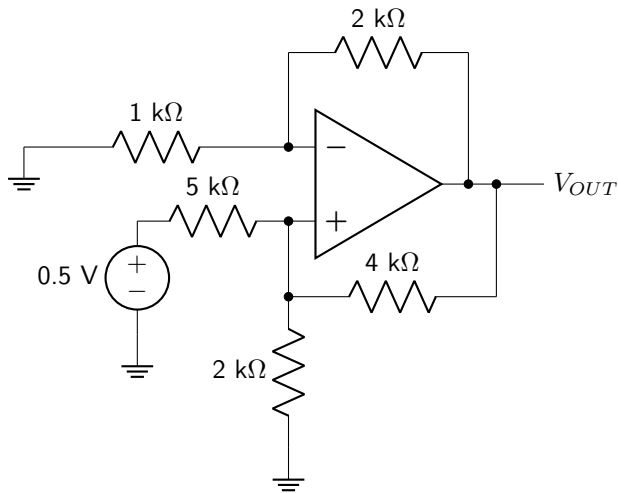


Figure 4.26: Circuit diagram for op-amps question 3.

4. Calculate the gain of the circuit shown in figure 4.27.

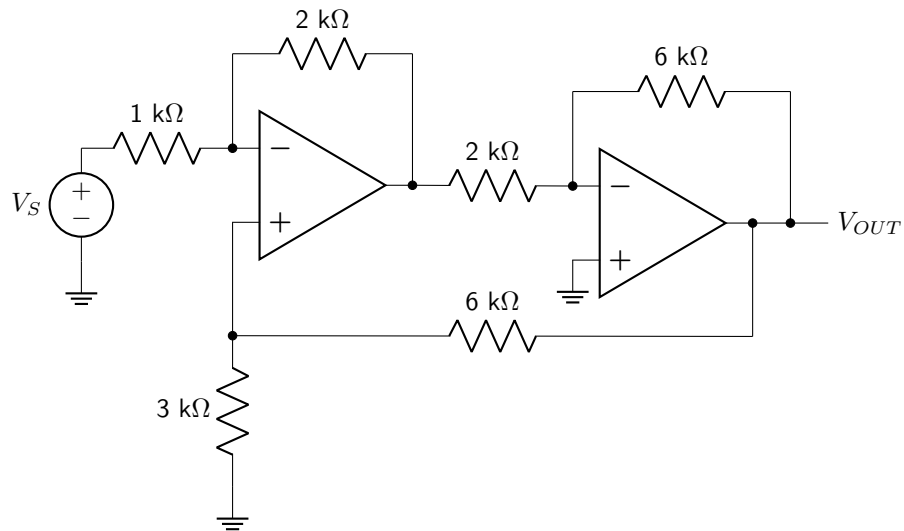


Figure 4.27: Circuit diagram for op-amps question 4.

5. Calculate the output voltage of the circuit shown in figure 4.28. (Assume that the supply voltage is sufficient to generate any output value.)

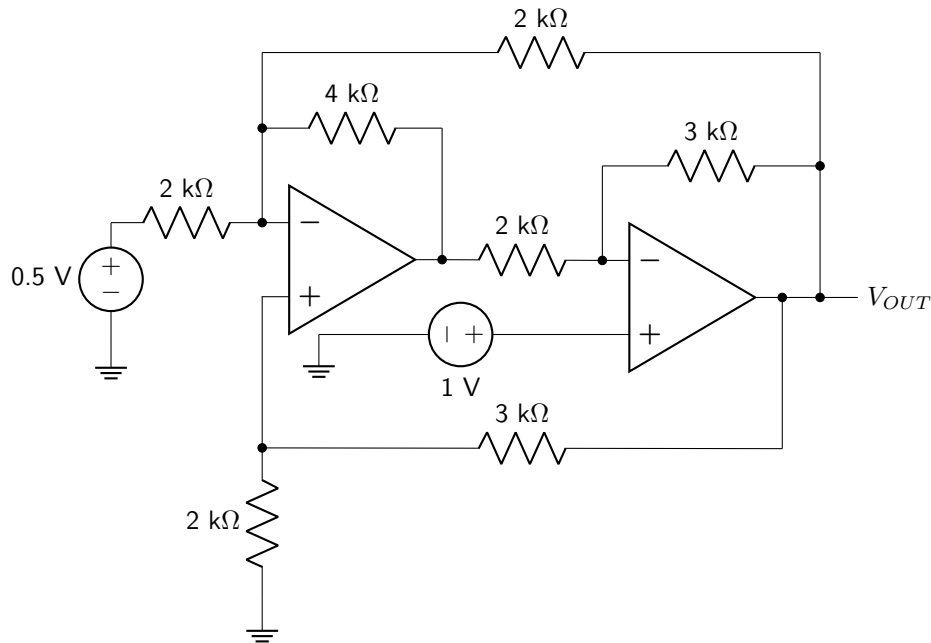


Figure 4.28: Circuit diagram for op-amps question 5.

5 Capacitors and RC Circuits

Capacitors are circuit elements that are capable of storing and releasing energy in the form of an electric field. The simplest model of a capacitor consists of two parallel plates of conducting material separated by an insulator (dielectric). As voltage is applied to the plates, charge builds up due to the fact that current cannot pass through the dielectric. This charge build-up creates a static electric field. This is depicted schematically in figure 5.1.

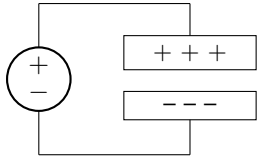


Figure 5.1: Two parallel plates of conducting material separated by an insulator will cause a buildup of charge when voltage is applied. This is called a capacitor.

The generic circuit symbol for a capacitor is shown in figure 5.2. It very closely resembles the parallel plate model.

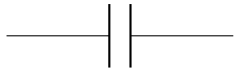


Figure 5.2: The circuit symbol for a capacitor.

Using the hydraulic analogy of electric circuits, a capacitor can be thought of as acting like a rubber membrane sealed in a pipe. When water flows through the pipe, water molecules cannot pass through the membrane, but the stretching causes current to flow nonetheless. You can think of current as flowing “through” a capacitor without any electrons passing from cathode to anode. Eventually, with the application of a high enough voltage (known as the breakdown voltage), electrons will be able to pass through the insulating layer (this is generally not desirable).

The amount of charge that can be stored in a capacitor depends on the applied voltage and a property of the capacitor known as capacitance. This relationship is shown in equation 5.1.

$$q(t) = v(t)C \quad (5.1)$$

Capacitance depends on the area of the plates (A), the distance between plates (d), and the permittivity of the dielectric material (ϵ) based on the mathematical relationship given in equation 5.2. The symbol for capacitance is C and the units are F (Farads).

$$C = \frac{\epsilon A}{d} \quad (5.2)$$

Dielectric permittivity describes how well a dielectric material can hold an electric charge. This value is

mathematically expressed as the product of the vacuum permittivity (ϵ_0) multiplied by the relative permittivity (ϵ_r), described in equation 5.3.

$$\epsilon = \epsilon_0 \epsilon_r \quad (5.3)$$

The value of the vacuum permittivity is given in equation 5.4.

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ F/m} \quad (5.4)$$

The relative permittivity is the ratio of the permittivity of a dielectric compared to a vacuum; the higher the value of ϵ_r , the better the dielectric is at holding an electric charge. The larger the relative dielectric permittivity, the higher the capacitance will be, based on equation 5.2.

Example: How big is one Farad?

To conceptualize how “big” one Farad is if created with a parallel plate capacitor, it can be calculated how large an area would be needed to create one Farad of capacitance given a separation of 1 mm between the two plates, with air being used as the dielectric material (air has a relative permittivity of one).

$$\begin{aligned} A &= \frac{Cd}{\epsilon} \\ &= \frac{(1 \text{ F})(1 \times 10^{-3} \text{ m})}{8.85 \times 10^{-12} \text{ F/m}} \\ &= 113,000,000 \text{ m}^2 \\ &= 113 \text{ km}^2 \end{aligned}$$

To put this size into context, a parallel plate capacitor using air as a dielectric with a spacing of 1 mm would completely engulf the city of Naperville, Illinois. This is clearly not the best way to build a capacitor!

5.1 Types of Capacitors

Capacitors can be categorized as non-polarized or polarized. In each of these groups, there are multiple different types of capacitors.

5.1.1 Non-Polarized Capacitors

Non-polarized capacitors, which have a circuit symbol shown in figure 5.2, can be connected in any orientation in a circuit. Generally speaking, there are two metal layers separated by a dielectric. The different types of non-polarized capacitors are named after the type of dielectric used (vacuum, air, glass, paper, mica, and ceramic, to name a few).

Ceramic capacitors are very frequently used due to their small size and low cost. Physically, they resemble

a small pancake with two metallic leads sticking out. Multiple layers of ceramic and metal are stacked up and placed into a package. The small size and dielectric choice means that while ceramic capacitors may have a large amount of capacitance per unit volume, their overall capacitance is low compared to other types of capacitors (namely, electrolytic capacitors). Because ceramic capacitors are non-polarized, they are good to use in AC circuits, particularly filter circuits.

Ceramic capacitors are stamped with a code that can be used to determine the value of the capacitance. A diagram of a ceramic capacitor, including code, is shown in figure 5.3.

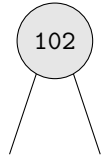


Figure 5.3: A schematic of a ceramic capacitor with numerical code. This particular capacitor would have a value of 1,000 pF.

The first two numbers act as the first two stripes on a resistor color code; they indicate the value of the capacitance. The third number represents the number of zeros following the value, in units of picofarads, or pF. For example, the 102 capacitor is 10 followed by two zeros: 1,000 pF.

Because ceramic capacitors are non-polarized, they are generally benign to use in circuits. The main hazard that can occur with a ceramic capacitor is to apply more voltage than the capacitor is rated for. At a certain point, the ceramic dielectric will physically break down. The effect of over-volting a ceramic capacitor can be cumulative (in other words, the capacitor may not catastrophically break down but will weaken and become less stable over time). Generally, when a ceramic capacitor is over-volted, it will fail open, which means that current will stop flowing once the capacitor is destroyed. This is a good thing in that it removes itself from the circuit when it beaks, but a bad thing in that it will stop working without necessarily being very obvious that it broke.

5.1.2 Variable Capacitors

Variable capacitors use two sets of parallel metal plates, each of which is connected to one of the leads. One set of parallel plates is held still, while the other set can rotate. When the moveable set of metal plates is rotated, the amount of overlap between the two sides of the metal will vary, which changes the value of the capacitance. A photograph of a rotary-style variable capacitor is shown in figure 5.4. On the left the metal plates have been rotated to have maximum overlap and capacitance. On the right the metal plates have been rotated to have minimum overlap and capacitance.

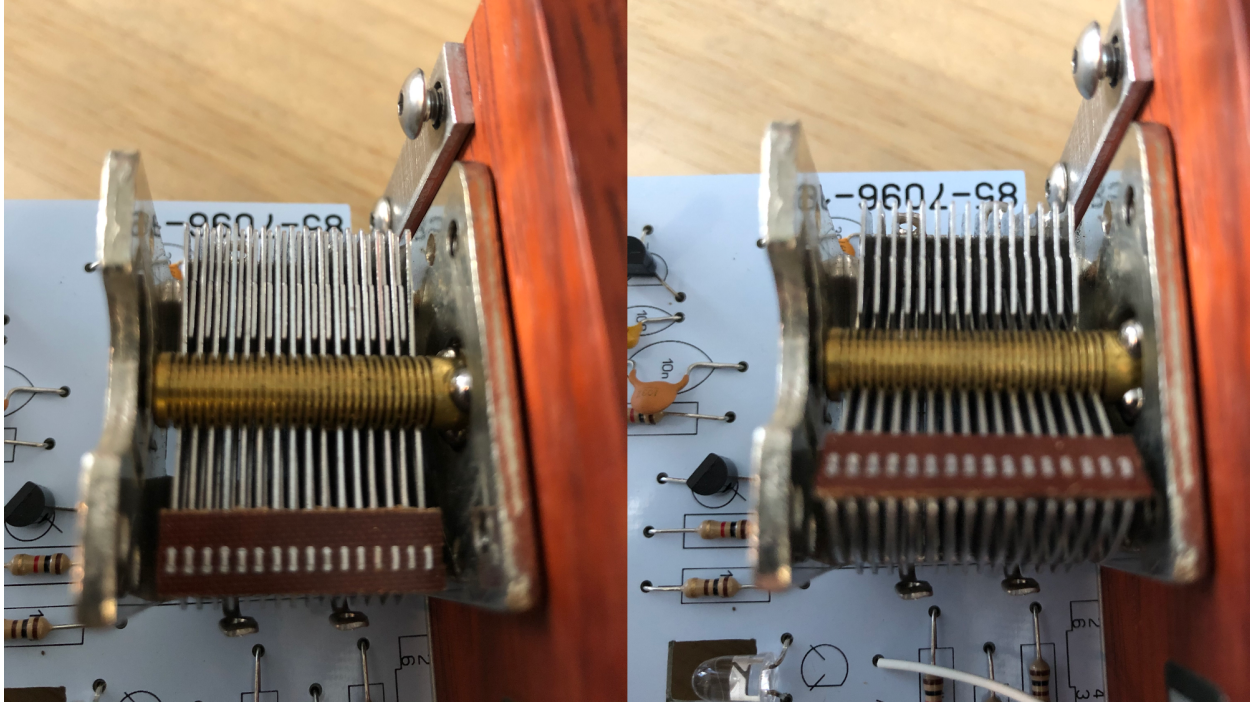


Figure 5.4: Photograph of a rotary-style variable capacitor with maximum metal plate overlap / capacitance (left) and minimum overlap / capacitance (right).

The circuit symbol for a variable capacitor is shown in figure 5.5.

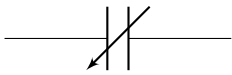


Figure 5.5: The circuit symbol for a variable capacitor.

Variable capacitors were used in radio circuits (before they became largely digital) to tune the correct radio frequency to amplify. The dial on the radio literally moved the rotating metal plates to change the capacitance and hence the circuit frequency.

5.1.3 Polarized Capacitors

Polarized capacitors are capacitors where one of the leads must be connected to high potential and the other to low potential. In other words: the capacitor cannot be connected in the circuit any which way. The circuit symbol for a polarized capacitor is shown in figure 5.6. The round end corresponds to the low potential side of the capacitor.

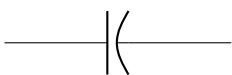


Figure 5.6: The circuit symbol for a polarized capacitor.

Polarized capacitors fall into two categories: electrolytic and supercapacitors. Supercapacitors will not

be discussed in this book except to note that they act as a hybrid between a capacitor and a rechargeable battery. Electrolytic capacitors are named after the type of metal used in the anode: aluminum, niobium, or tantalum. Each of the anodes is roughened (which acts to increase the surface area of the capacitor) and oxidized (creating a very thin dielectric layer, decreasing the distance between cathode and anode). An electrolyte gel acts as the cathode. Electrolytic capacitors generally have a very large amount of capacitance. However, because they are polarized, they cannot be used in all circuit applications and are mostly used in DC circuits.

Aluminum electrolytic capacitors have an aluminum anode. They tend to be inexpensive and easy to obtain. Aluminum electrolytic capacitors are packaged in a cylinder. In a radial configuration, one of the leads is shorter than the other and is situated near a stripe on the cylindrical package. The short lead / stripe indicates the low potential end of the capacitor. This is depicted in figure 5.7.

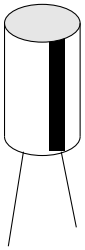


Figure 5.7: Diagram of an aluminum electrolytic capacitor. The short lead (also has a stripe on the casing) corresponds to the low potential connection.

The top of the cylindrical packaging is typically scored to allow breakage in case the capacitor is reverse-biased (plugged in backwards) or over-volted. On top of the hazard of over-volting the capacitor, reverse biasing the capacitor is also problematic. Aluminum electrolytic capacitors can and will explode under the right circumstances, and will release anisotropically via the scored top of the cylinder. (Therefore, it is not a good idea to point the top of an aluminum electrolytic capacitor at anything important, especially your face, if it is uncertain if it's connected properly or with the correct amount of voltage). Thankfully, as with ceramic capacitors, an aluminum electrolytic capacitor will fail open (removing itself from the circuit after exploding). The amount of capacitance of an aluminum electrolytic capacitor, as well as the voltage rating, is printed on the packaging.

Tantalum electrolytic capacitors are made of a tantalum anode. They have a very large capacitance per unit of volume, making them good to use in applications where a small footprint is required. Physically, tantalum electrolytic capacitors resemble the pancake look of a ceramic capacitor. However, tantalum electrolytic capacitors are polarized and must be treated accordingly. Usually, the high potential end (anode) is indicated on the packaging with a plus sign, as shown in figure 5.8.

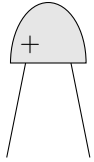


Figure 5.8: A diagram of a tantalum electrolytic capacitor. The plus sign corresponds to the high potential lead.

Tantalum capacitors have the property of failing short when over-volted. This can be very dangerous, as a short exacerbates the problem and can lead to a fire. Therefore it is very important to respect the polarity and voltage rating of a tantalum electrolytic capacitor.

5.2 Measuring Capacitance

Capacitors can be measured with a digital multimeter or with an LRC meter. A digital multimeter will charge a capacitor with a known current and measure the voltage to calculate the capacitance. An LRC meter applies an AC voltage, measures the AC current, and uses the phase and amplitude differences to calculate the capacitance.

Capacitance must be measured from one end of the capacitor (or equivalent capacitance) to the other, in parallel, as shown in figure 5.9.

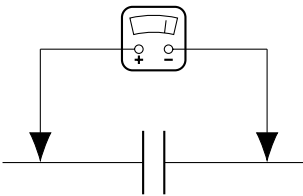


Figure 5.9: A diagram of how capacitance is measured in parallel (across a component or components).

It is important to ensure that the capacitor(s) are completely discharged before making a measurement. Capacitors can be discharged by shorting the leads with a high value, high-wattage resistor for several seconds.

5.3 Voltage-Current Relationship in a Capacitor

The relationship between voltage and current in a capacitor can be quantified using the relationship between capacitance and charge (equation 5.1) as well as the relationship between current and charge (equation 1.1). This relationship is shown in terms of current in equation 5.5 and derived below.

$$i(t) = C \frac{dv(t)}{dt} \quad (5.5)$$

DERIVATION

$$\begin{aligned}
 i(t) &= \frac{dq(t)}{dt} \\
 &= \frac{d(v(t)C)}{dt} \\
 &= C \frac{dv(t)}{dt}
 \end{aligned}$$

Based on equation 5.5, if the voltage remains unchanging through time, there will be no current flowing through the branch that contains a capacitor. Therefore, in DC conditions, a capacitor can be replaced by an open circuit. This analysis technique will be used frequently in this book to explore the transient response of capacitor circuits.

This equation can be refigured to solve for the voltage drop over a capacitor, shown in equation 5.6.

$$v(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau \quad (5.6)$$

Although calculus is required to fully analyze the relationship between voltage and current in a capacitor, the relationship between the two variables is linear. This means that capacitors are linear circuit elements, and all of the techniques discussed in this book (such as superposition and Thévenin equivalence, for example) are still relevant with capacitive circuits.

5.4 Equivalent Capacitance

Just as it was possible to combine resistors together to find an equivalent resistance, it is possible to combine capacitors together to find an equivalent capacitance. The rules (and mathematical relationships) for calculating series and parallel combinations of capacitors are explained below.

5.4.1 Series Combinations of Capacitors

Capacitors in series all share a common branch, as shown in figure 5.10. Each individual voltage drop is shown (numbered with subscripts), as well as the total voltage drop over the combination of capacitors ($v(t)$). The current flow through all of the capacitors is the same and is labeled $i(t)$.

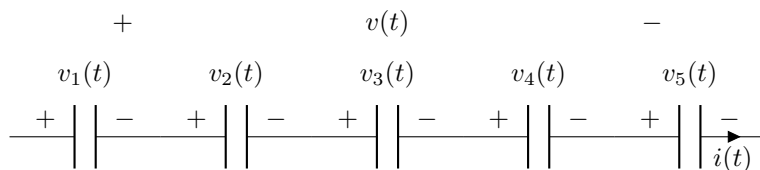


Figure 5.10: Five capacitors in series in a circuit.

As $v(t)$ is equal to the sum of each individual voltage drop, the current-voltage relationship in capacitors

(equation 5.6) can be used to determine the equivalent capacitance of series combinations of capacitors. This is shown in equation 5.7 and derived below.

$$\frac{1}{C_{EQ}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots + \frac{1}{C_n} \quad (5.7)$$

DERIVATION

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) + v_3(t) + \dots + v_n(t) \\ \int_{-\infty}^t \frac{1}{C_{EQ}} i(\tau) d\tau &= \int_{-\infty}^t \frac{1}{C_1} i(\tau) d\tau + \int_{-\infty}^t \frac{1}{C_2} i(\tau) d\tau + \int_{-\infty}^t \frac{1}{C_3} i(\tau) d\tau + \dots + \int_{-\infty}^t \frac{1}{C_n} i(\tau) d\tau \\ \frac{1}{C_{EQ}} &= \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots + \frac{1}{C_n} \end{aligned}$$

Capacitors in series therefore combine similarly to resistors in parallel. The equivalent capacitance cannot be larger than the smallest capacitor in the series combination.

5.4.2 Parallel Combinations of Capacitors

Capacitors in parallel all share two nodes, as shown in figure 5.11. Each individual current flow is shown (numbered with subscripts), as well as the total current flow through the combination of capacitors ($i(t)$). The voltage drop over all of the capacitors is the same and is labeled $v(t)$.

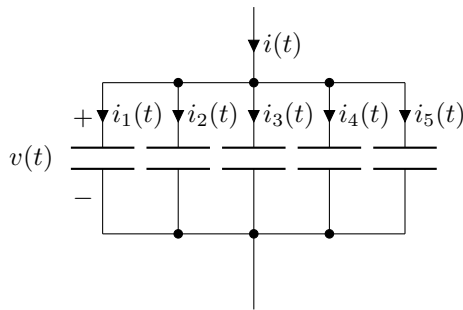


Figure 5.11: Five capacitors in parallel in a circuit.

As $i(t)$ is equal to the sum of each individual current flow, the current-voltage relationship in capacitors (equation 5.5) can be used to determine the equivalent capacitance of parallel combinations of capacitors. This is shown in equation 5.8 and derived below.

$$C_{EQ} = C_1 + C_2 + C_3 + \dots + C_n \quad (5.8)$$

DERIVATION

$$i(t) = i_1(t) + i_2(t) + i_3(t) + \dots + i_n(t)$$

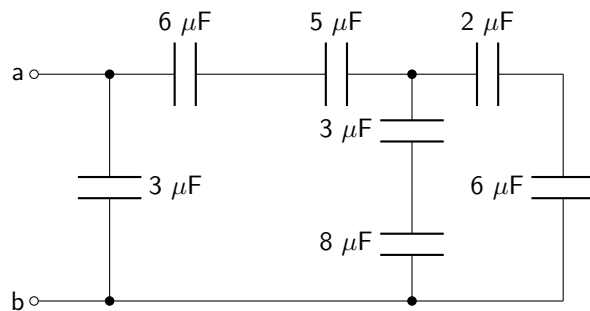
$$C_{EQ} \frac{dv(t)}{dt} = C_1 \frac{dv(t)}{dt} + C_2 \frac{dv(t)}{dt} + C_3 \frac{dv(t)}{dt} + \dots + C_n \frac{dv(t)}{dt}$$

$$C_{EQ} = C_1 + C_2 + C_3 + \dots + C_n$$

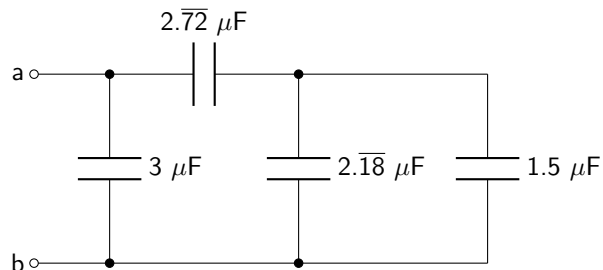
Capacitors in parallel therefore combine similarly to resistors in series. The equivalent capacitance cannot be smaller than the largest capacitor in the parallel combination.

Example: Calculating equivalent capacitance

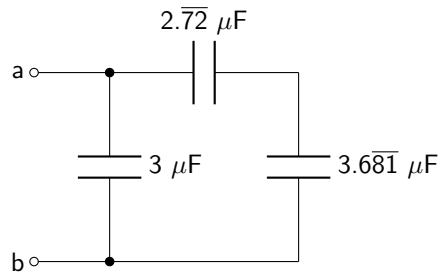
Calculate the equivalent capacitance between nodes a and b.



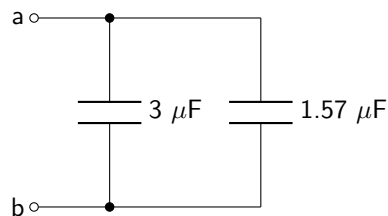
The $2 \mu\text{F}$ and $6 \mu\text{F}$ are in series and can be combined together. This combination is $1.5 \mu\text{F}$. Similarly, the $3 \mu\text{F}$ is in series with the $8 \mu\text{F}$ with an equivalent capacitance of $2.\overline{18} \mu\text{F}$. Last, the $6 \mu\text{F}$ and $5 \mu\text{F}$ capacitors are in series with an equivalent capacitance of $2.\overline{72} \mu\text{F}$. The circuit can now be re-drawn.



At this point the $1.5 \mu\text{F}$ and $2.\overline{18} \mu\text{F}$ capacitors can be combined in parallel to obtain $3.\overline{681} \mu\text{F}$. The circuit can again be re-drawn.



The $2.72 \mu\text{F}$ capacitor is in series with the $3.681 \mu\text{F}$ capacitor. Combined, their equivalent capacitance is $1.57 \mu\text{F}$. The circuit can be re-drawn again.



The equivalent capacitance can now be calculated by summing together the two capacitances. C_{EQ} is $4.57 \mu\text{F}$.

5.5 Differentiator and Integrator Circuits

The current-voltage relationship in a capacitor can be used to create circuits capable of calculating the derivative or integral of a signal.

First, consider the circuit shown in figure 5.12. This is a differentiator circuit. The output voltage is directly proportional to the derivative of the input voltage.

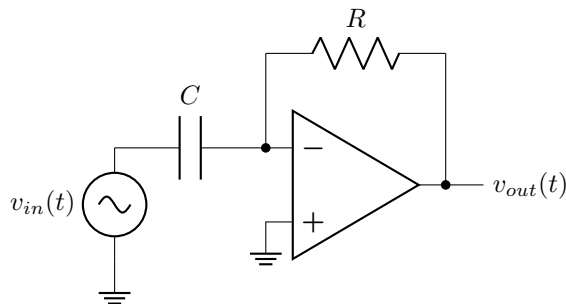


Figure 5.12: Op-amp differentiator circuit.

The current flowing through the capacitor can be described by equation 5.5. Assume the direction of current is from left to right in both the resistor and the capacitor. The voltage drop over the capacitor is therefore $v_{in}(t)$. The output voltage of the circuit can then be described in equation 5.9, which is also

derived below.

$$v_{out}(t) = -RC \frac{d}{dt} v_{in}(t) \quad (5.9)$$

DERIVATION

$$\begin{aligned} C \frac{d}{dt} v_{in}(t) &= -\frac{v_{out}(t)}{R} \\ -RC \frac{d}{dt} v_{in}(t) &= v_{out}(t) \end{aligned}$$

Next, consider the circuit shown in figure 5.13. This is an integrator circuit. The output voltage is directly proportional to the integral of the input voltage.

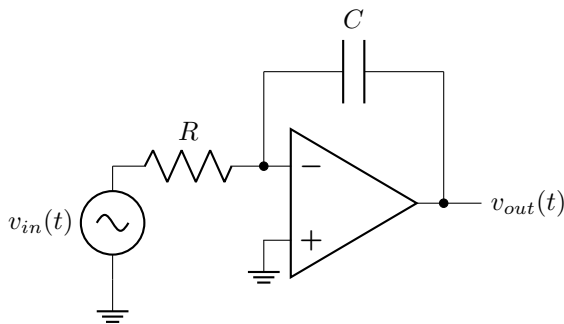


Figure 5.13: Op-amp integrator circuit.

The current flowing through the capacitor can be described by equation 5.5. Assume the direction of current is from left to right in both the resistor and the capacitor. The voltage drop over the capacitor is therefore $-v_{out}(t)$. The output voltage of the circuit can then be described in equation 5.10, which is also derived below.

$$v_{out}(t) = -\frac{1}{RC} \int v_{in}(t) dt \quad (5.10)$$

DERIVATION

$$\begin{aligned} \frac{v_{in}(t)}{R} &= C \frac{d}{dt} [-v_{out}(t)] \\ \int \frac{v_{in}(t)}{R} dt &= \int C \frac{d}{dt} [-v_{out}(t)] dt \\ \frac{1}{R} \int v_{in}(t) dt &= -C v_{out}(t) \\ -\frac{1}{CR} \int v_{in}(t) dt &= v_{out}(t) \end{aligned}$$

5.6 Resistor-Capacitor (RC) Circuits

Resistor-capacitor (RC) circuits are circuits that contain one capacitor (or combinations of capacitors that can be reduced down to one equivalent capacitance) as well as at least one resistive element. In this chapter, the analysis of these circuits will be performed when DC conditions are changed suddenly to another set of DC conditions. For example: a capacitor will have been subjected to a constant initial voltage for a long period of time, and then suddenly disconnected from a circuit. Or: a capacitor will have been discharged for a long period of time before suddenly being connected to a circuit with a DC voltage. This short-term change in voltage and current is known as a transient analysis.

Resistors, by themselves, are incapable of having a transient change. This is because they do not store energy. Because a capacitor stores energy, it is able to experience a transient change in response to changing DC conditions of the circuit.

An important property of capacitors that will aid in the analysis of the transient response is that **the voltage over a capacitor cannot change instantaneously**. That is: $v(0^-) = v(0^+)$. In an ideal analysis (which is what this book will use) where capacitors have no parasitic effects (stray inductance, for example), the current through a capacitor can change instantaneously and therefore be discontinuous in time.

5.6.1 Discharging Transient Response

The discharging response is a special case of the transient response that specifically looks at what happens when a capacitor is initially connected to a DC voltage that is suddenly removed from a circuit. In other words: the capacitor, after being disconnected from the power supply, discharges its stored energy through the resistive element(s) of the circuit. The circuit diagram shown in figure 5.14 will be used to analyze this discharging response. The switch is in position a for a long time before suddenly moving to position b at time $t = 0$.

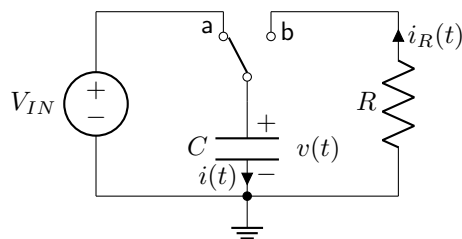


Figure 5.14: An RC circuit used to analyze the discharging transient response.

Kirchhoff's laws can be used to analyze the circuit in the initial condition. Because the only two elements are the voltage source and the capacitor, and they are in parallel, the initial voltage drop over the capacitor must be equal to the source voltage. In other words: $v(0) = V_{IN}$.

Kirchhoff's laws are also used to analyze the circuit in the discharging condition, once the switch has been flipped to position b. The current flowing through the capacitor ($i(t)$) is equal to $i_R(t)$ as shown

in figure 5.14. The voltage-current relationship in a capacitor (equation 5.5), as well as Ohm's law over the resistor ($i_R(t) = -v(t)/R$), can be used to determine the discharging transient response, shown in equation 5.11.

$$0 = \frac{dv(t)}{dt} + \frac{1}{RC}v(t) \quad (5.11)$$

The equation for the voltage drop in a transient discharging analysis is a first order, linear, ordinary differential equation. This is consistent with the fact that the circuit itself is a first order, linear circuit! The solution to the differential equation will not be derived in this book but is given in equation 5.12.

$$v(t) = v(0^+)e^{-\frac{t}{RC}}u(t) + v(0^-)u(-t) \quad (5.12)$$

Using equation 5.5, the transient response of current flowing through the discharging circuit can also be determined. This is derived in equation 5.13. Note that as the capacitor acts as an open in DC steady-state conditions, the current through the circuit will be zero until $t = 0^+$.

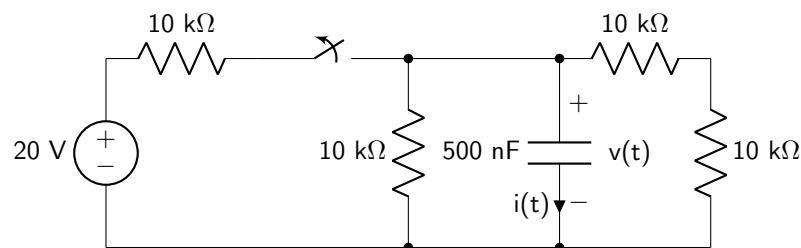
$$i(t) = \frac{-v(0^+)}{R}e^{-\frac{t}{RC}}u(t) \quad (5.13)$$

The quantity RC defines the quickness that the circuit discharges to zero and is known as the RC time constant. The mathematical symbol for the RC time constant is τ . Specifically, τ defines the amount of time it takes the voltage to decay to 36.8% of its maximum value.

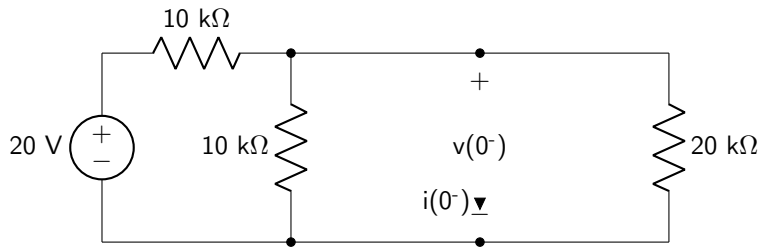
If there is more than one resistive element in the discharging circuit, then the value of R in the time constant is equal to the equivalent resistance as seen by the capacitor.

Example: Discharging transient response

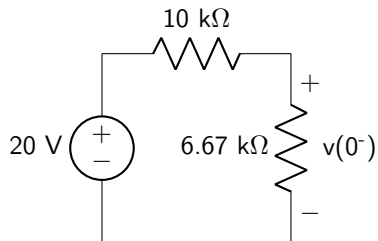
Calculate $v(t)$ and $i(t)$ for the circuit below. The switch opens at a time of 0 seconds.



The initial conditions can be calculated. Re-draw the circuit to represent the time before the switch opens. Because the circuit is under DC conditions, the capacitor can be represented as an open. The two $10\text{ k}\Omega$ resistors can be combined in series.



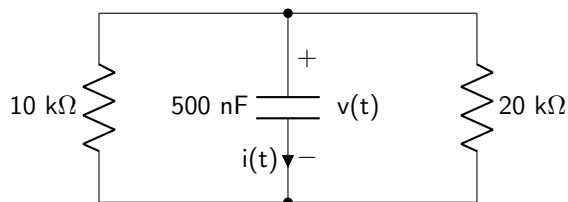
The initial current $i(0^-)$ is zero due to the presence of an open circuit. The $20\text{ k}\Omega$ and $10\text{ k}\Omega$ resistors are in parallel with each other. The circuit can be re-drawn.



Use a voltage divider to calculate $v(0^-)$.

$$\begin{aligned} v(0^-) &= 20\text{ V} \left(\frac{6.67\text{ k}\Omega}{16.67\text{ k}\Omega} \right) \\ &= 8\text{ V} \end{aligned}$$

Analyze the discharging circuit. Re-draw for time $t \geq 0$. The two $10\text{ k}\Omega$ resistors that are in series have been combined to make the circuit simpler.



The equivalent resistance seen by the capacitor is $10\text{ k}\Omega$ in parallel with $20\text{ k}\Omega$, or $6.67\text{ k}\Omega$. This can be used to calculate the RC time constant.

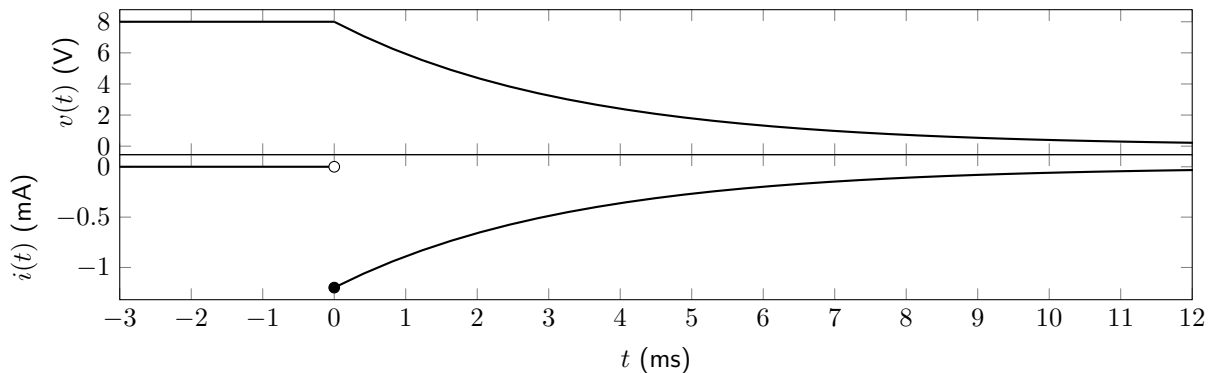
$$\begin{aligned} \tau &= (500\text{ nF})(6.67\text{ k}\Omega) \\ &= 0.003\text{ s} \end{aligned}$$

The equations for $v(t)$ and $i(t)$ can now be derived using equations 5.12 and 5.13.

$$v(t) = 8 \text{ V } e^{-300t} u(t) + 8 \text{ V } u(-t)$$

$$i(t) = -1.2 \text{ mA } e^{-300t} u(t)$$

These two signals are plotted below as functions of time.



5.6.2 Charging Transient (Step) Response

The charging transient (step) response of a circuit corresponds to how the voltage and/or current changes with respect to a sudden application of voltage to the circuit components. This is another specific case of the general RC transient response, which will be discussed in section 5.6.3. Specifically, the charging transient response refers to a fully discharged capacitor suddenly being connected to a circuit containing at least one resistive element and a power supply. The circuit diagram shown in figure 5.15 will be used to analyze the charging response. The switch is in position b for a long time before suddenly moving to position a at time $t = 0$.

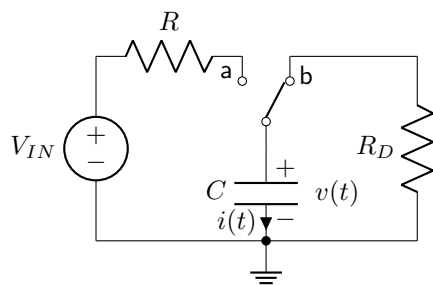


Figure 5.15: An RC circuit used to analyze the charging transient (step) response.

In the initial state of the circuit, all of the energy that may have been stored in the capacitor has completely discharged through the discharging resistor (R_D) giving an initial voltage drop of 0 V.

In the charging condition (when the switch has been moved to position a), Kirchhoff's voltage law can be used to analyze the voltage drop over each circuit element. From this, the differential equation for $v(t)$

can be derived. This is shown in equation 5.14 and is derived below. The term $v(\infty)$ corresponds to the final, long-term value of the voltage dropped over the capacitor. It will be equal to V_{IN} in the case of the circuit shown in figure 5.15.

$$0 = \frac{dv(t)}{dt} + \frac{v(t) - v(\infty)}{RC} \quad (5.14)$$

DERIVATION

$$\begin{aligned} 0 &= -v(\infty) + Ri(t) + v(t) \\ &= -v(\infty) + RC \frac{dv(t)}{dt} + v(t) \\ &= \frac{dv(t)}{dt} + \frac{v(t) - v(\infty)}{RC} \end{aligned}$$

The solution to this differential equation (assuming that the initial voltage drop over the capacitor is 0 V) is given in equation 5.15.

$$v(t) = v(\infty) \left[1 - e^{-\frac{t}{RC}} \right] u(t) \quad (5.15)$$

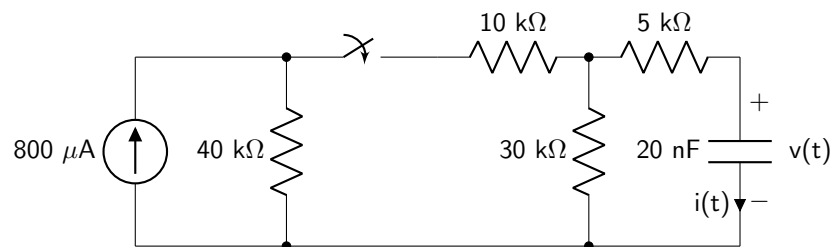
The value of the RC time constant, τ , is still equal to the product of equivalent capacitance and equivalent resistance as seen by the capacitor. The time constant in a charging circuit dictates how much time it takes the circuit to charge up to 63.2% of the maximum voltage value.

Equation 5.5 can be used to find the current through the capacitor during the transient charging process. The result is shown in equation 5.16.

$$i(t) = \frac{v(\infty)}{R} e^{-\frac{t}{RC}} u(t) \quad (5.16)$$

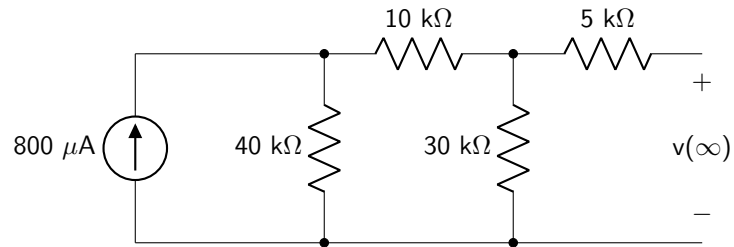
Example: Charging transient response

Calculate $v(t)$ and $i(t)$ for the circuit below. The switch closes at a time of 0 seconds.

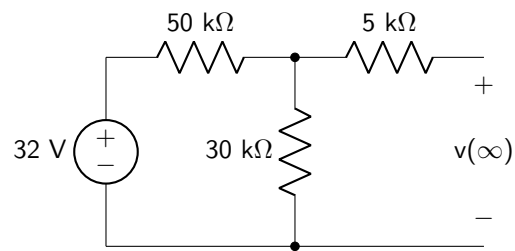


The final (steady-state) conditions can be calculated. Re-draw the circuit to represent the time well after the switch closes. Because the circuit is under DC conditions, the capacitor can be represented as an open.

The final value of the current will be 0 mA.



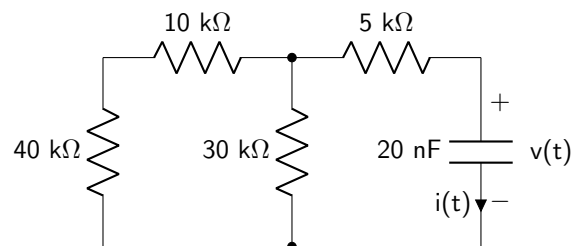
Source transformation can be used to calculate the steady-state value of the voltage. The 40 kΩ resistor can also be added to the 10 kΩ resistor, as they will be in series after the source transformation.



Use a voltage divider to calculate $v(\infty)$.

$$\begin{aligned} v(\infty) &= 32 \text{ V} \left(\frac{30 \text{ k}\Omega}{80 \text{ k}\Omega} \right) \\ &= 12 \text{ V} \end{aligned}$$

Analyze the charging circuit. Re-draw for time $t \geq 0$. Deactivate the current source by replacing it with an open to calculate the equivalent resistance as seen by the capacitor.



The equivalent resistance is calculated below.

$$\begin{aligned} R_{EQ} &= (40 \text{ k}\Omega + 10 \text{ k}\Omega) // 30 \text{ k}\Omega + 5 \text{ k}\Omega \\ &= 50 \text{ k}\Omega // 30 \text{ k}\Omega + 5 \text{ k}\Omega \\ &= 18.75 \text{ k}\Omega + 5 \text{ k}\Omega \\ &= 23.75 \text{ k}\Omega \end{aligned}$$

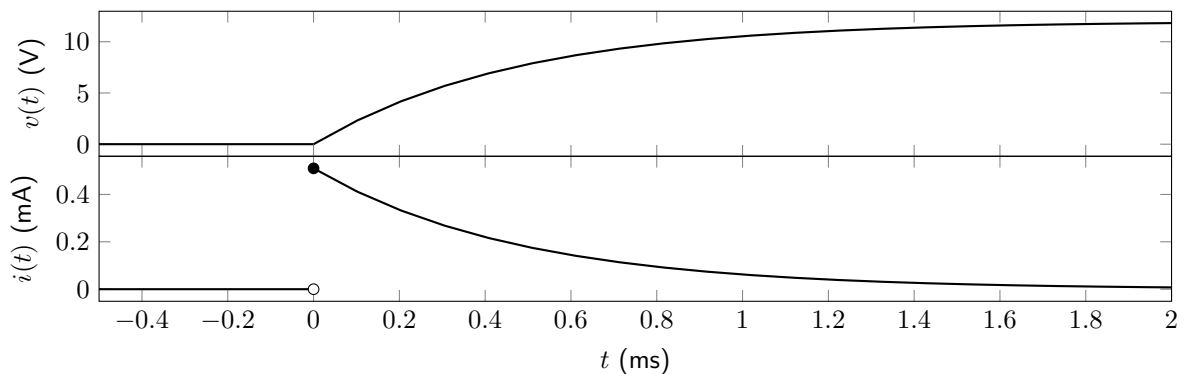
Calculate the RC time constant.

$$\begin{aligned} \tau &= (20 \text{ nF})(23.75 \text{ k}\Omega) \\ &= 4.64 \times 10^{-4} \text{ s} \end{aligned}$$

The equations for $v(t)$ and $i(t)$ can now be derived using equations 5.15 and 5.16.

$$\begin{aligned} v(t) &= [12 \text{ V} - 12 \text{ V} e^{-2105t}] u(t) \\ i(t) &= 0.51 \text{ mA} e^{-2105t} u(t) \end{aligned}$$

These two signals are plotted below as functions of time.



5.6.3 General Transient Response

Two special cases of the RC transient response have been analyzed: the discharging response ($v(\infty) = 0$) and the step response ($v(0) = 0$). In general, the initial voltage drop over a capacitor and the final value of the capacitor can be any arbitrary values. The most general form of the equation for the transient response of an RC circuit is therefore given in equation 5.17.

$$v(t) = \left[v(\infty) + (v(0^+) - v(\infty)) e^{-\frac{t}{RC}} \right] u(t) + v(0^-) u(-t) \quad (5.17)$$

Current flow through the capacitor can be calculated by using equation 5.5. This is shown in equa-

tion 5.18. Note that initial current flow will be zero due to the DC steady-state. This will not change until $t = 0^+$.

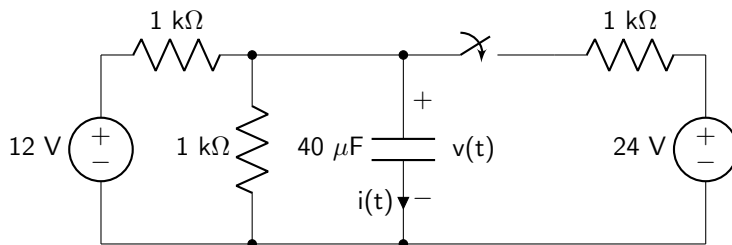
$$i(t) = \frac{v(\infty) - v(0^+)}{R} e^{-\frac{t}{RC}} u(t) \quad (5.18)$$

In the general transient response, the RC time constant refers to how much time it takes for the voltage drop over the capacitor to reach a value given by equation 5.19.

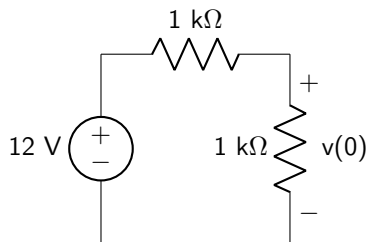
$$v(\tau) = 0.632v(\infty) + 0.368v(0^+) \quad (5.19)$$

Example: General transient response

Calculate $v(t)$ and $i(t)$ for the circuit below. The switch closes at a time of 0 seconds.



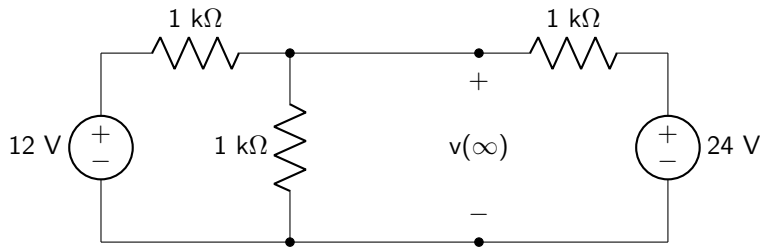
Analyze the initial conditions of the circuit. The initial value of the current $i(0^-)$ is zero.



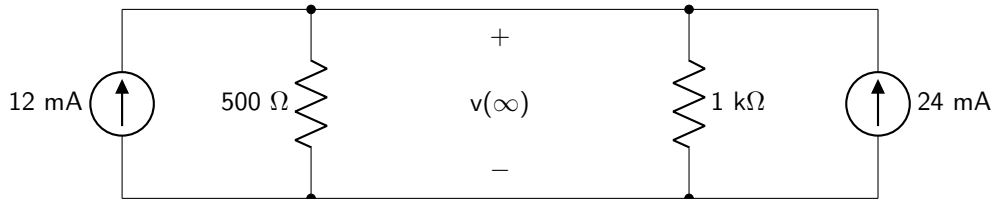
Use a voltage divider to find the initial voltage drop over the capacitor.

$$\begin{aligned} v(0) &= 12 \text{ V} \left(\frac{1 \text{ k}\Omega}{2 \text{ k}\Omega} \right) \\ &= 6 \text{ V} \end{aligned}$$

Analyze the final (steady-state) conditions of the circuit. The steady-state value of the current will be zero.



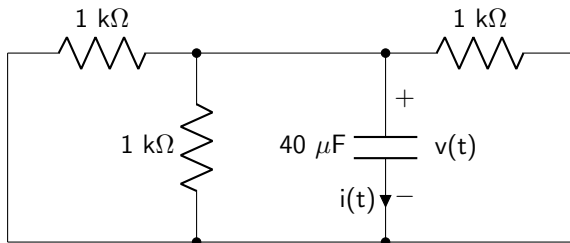
Source transformation can be used to simplify this circuit.



Use Ohm's law to calculate $v(\infty)$.

$$\begin{aligned} v(\infty) &= 36 \text{ mA}(0.5 \text{ k}\Omega // 1 \text{ k}\Omega) \\ &= 12 \text{ V} \end{aligned}$$

Analyze the circuit for time $t \geq 0$ and deactivate all power sources to determine the equivalent resistance as seen by the capacitor. The circuit is re-drawn below.



The equivalent resistance is calculated below.

$$\begin{aligned} R_{EQ} &= 1 \text{ k}\Omega // 1 \text{ k}\Omega // 1 \text{ k}\Omega \\ &= 333.33 \Omega \end{aligned}$$

Calculate the RC time constant.

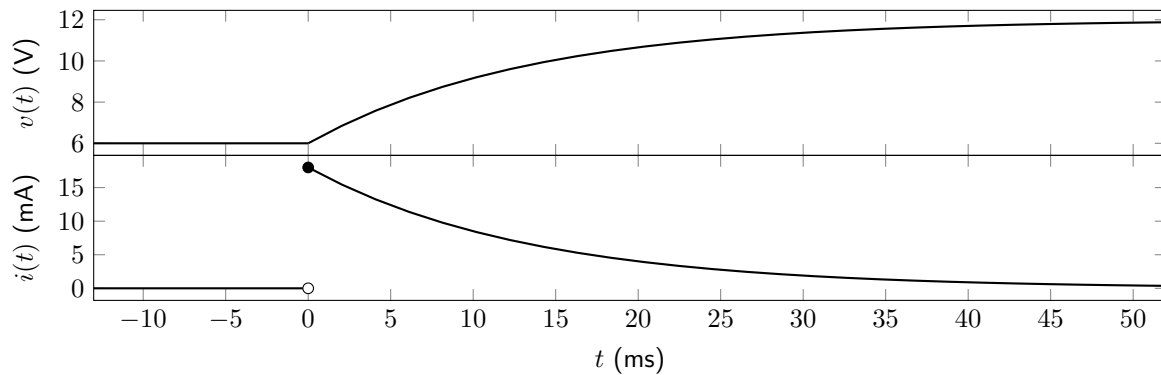
$$\begin{aligned} \tau &= (40 \mu\text{F})(333.33 \Omega) \\ &= 0.013 \text{ s} \end{aligned}$$

The equations for $v(t)$ and $i(t)$ can now be derived using equations 5.17 and 5.18.

$$v(t) = [12 \text{ V} - 6 \text{ V} e^{-75t}] u(t) + 6 \text{ V} u(-t)$$

$$i(t) = 18 \text{ mA} e^{-75t} u(t)$$

These two signals are plotted below as functions of time.



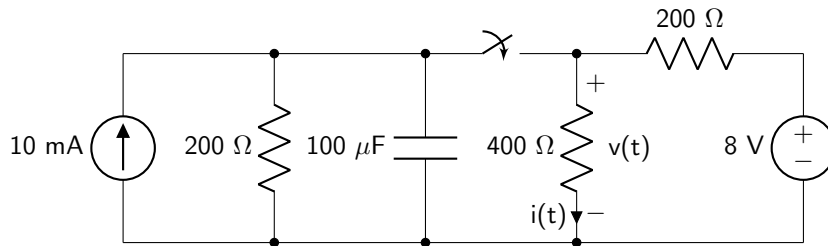
All of the equations in this section of the text have discussed finding the voltage drop over the capacitor. It is also possible to derive a first order equation to find voltage over or current through any arbitrary component in a circuit. It is important to note that the initial conditions may be discontinuous. This is because only the voltage dropped over a capacitor is going to be necessarily the same at time $t = 0^-$ and $t = 0^+$. All other quantities must be analyzed at $t = 0^-$ (steady state initial conditions), and again at exactly $t = 0$ to determine the response at $t = 0^+$. Then, final steady-state conditions can be analyzed to find the response at $t \rightarrow \infty$. If the initial condition is discontinuous, the value at $t = 0^+$ should be used in the equation.

The steps for deriving an equation for the general transient response for an RC circuit follow. To reiterate, because the discharging and charging responses are just special cases of the general transient response, these steps can be followed to complete any RC circuit transient analysis.

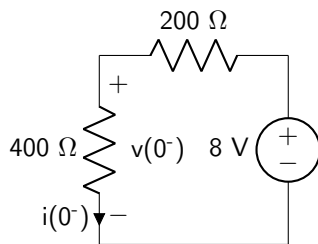
1. Calculate the initial condition(s) of the circuit (any voltage(s) or current(s) that you are asked to find).
2. If necessary, calculate any needed circuit properties at time $t = 0^+$.
3. Calculate the final condition(s) of the circuit.
4. Use the final configuration of the circuit to calculate the equivalent resistance as seen by the capacitor to calculate the RC time constant.

Example: General transient response

Calculate $v(t)$ and $i(t)$ for the circuit below. The switch closes at a time of 0 seconds.



Analyze the initial conditions of the circuit. These are valid up until $t = 0^-$. It is also useful to note that the initial voltage drop over the capacitor (which must be continuous at 0 seconds) will be 2 V (10 mA times 200 Ω).



Use a voltage divider to find the initial voltage drop over the resistor.

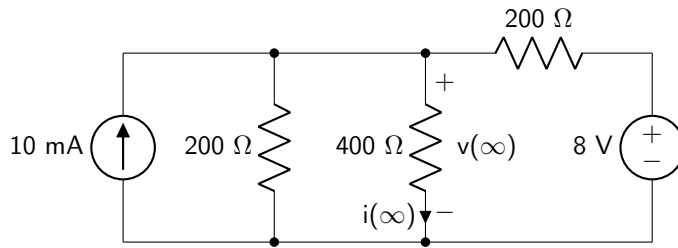
$$\begin{aligned} v(0^-) &= 8 \text{ V} \left(\frac{400 \Omega}{600 \Omega} \right) \\ &= 5.33 \text{ V} \end{aligned}$$

Ohm's law can be used to calculate the initial current flow through the resistor.

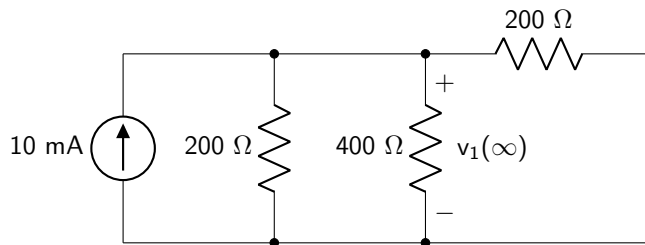
$$\begin{aligned} i(0^-) &= \frac{5.33 \text{ V}}{0.4 \text{ k}\Omega} \\ &= 13.33 \text{ mA} \end{aligned}$$

Because the voltage drop over and current flow through the resistor will not be continuous when the switch closes, the circuit needs to be analyzed at $t = 0^+$. Because the resistor will be in parallel with the capacitor, the voltage drop at that moment will be the same as the initial voltage drop over the capacitor: 2 V. Ohm's law can be used to calculate $i(0^+)$ which is 5 mA.

Next, the steady-state conditions of the circuit should be found to determine $v(\infty)$ and $i(\infty)$. The capacitor can be replaced by an open.

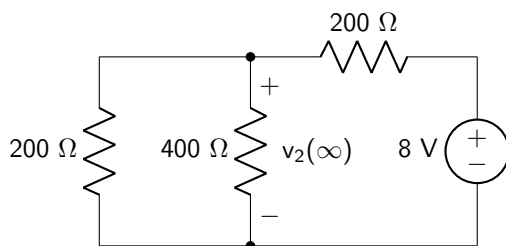


Superposition can be used to solve this circuit.



Use Ohm's law to calculate $v_1(\infty)$.

$$\begin{aligned} v_1(\infty) &= (10 \text{ mA})(0.2 \text{ k}\Omega // 0.2 \text{ k}\Omega // 0.4 \text{ k}\Omega) \\ &= (10 \text{ mA})(0.08 \text{ k}\Omega) \\ &= 0.8 \text{ V} \end{aligned}$$

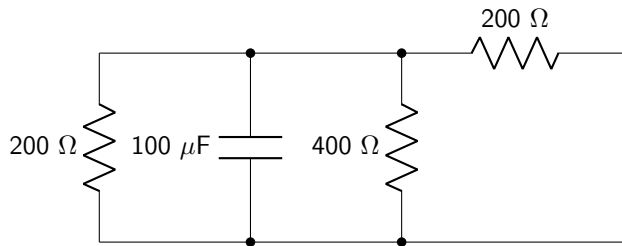


Use a voltage divider to find $v_2(\infty)$.

$$\begin{aligned} v_2(\infty) &= 8 \text{ V} \left(\frac{133.3 \Omega}{333.3 \Omega} \right) \\ &= 3.2 \text{ V} \end{aligned}$$

The total voltage drop over the resistor in steady-state is therefore 4 V. Using Ohm's law, $i(\infty)$ is 10 mA.

Analyze the circuit for time $t \geq 0$ and deactivate all power sources to determine the equivalent resistance as seen by the capacitor. This is to find the RC time constant τ . The circuit is re-drawn below.



The equivalent resistance is calculated below.

$$R_{EQ} = 0.2 \text{ k}\Omega // 0.2 \text{ k}\Omega // 0.4 \text{ k}\Omega$$

$$= 80 \text{ }\Omega$$

Calculate the RC time constant.

$$\tau = (100 \text{ }\mu\text{F})(80 \text{ }\Omega)$$

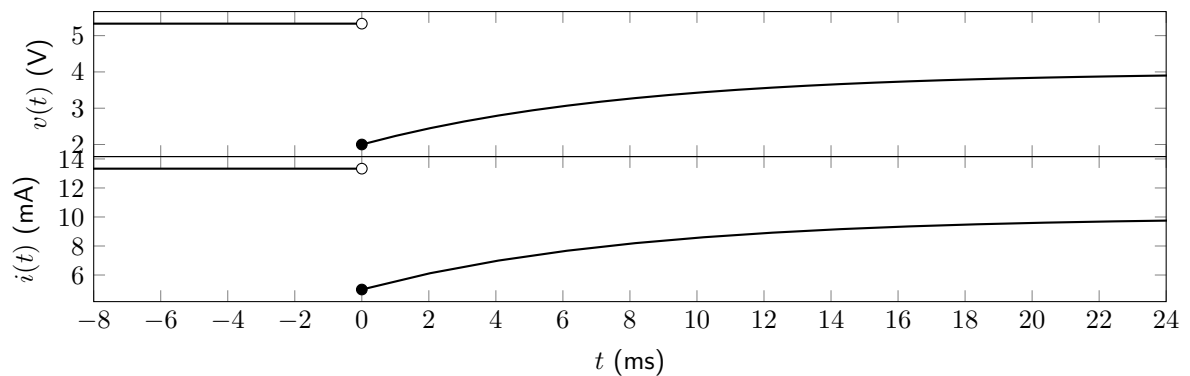
$$= 0.008 \text{ s}$$

The equations for $v(t)$ and $i(t)$ can now be derived using equations 5.17 and Ohm's law.

$$v(t) = [4 \text{ V} - 2 \text{ V} e^{-125t}] u(t) + 5.33 \text{ V} u(-t)$$

$$i(t) = [10 \text{ mA} - 5 \text{ mA} e^{-125t}] u(t) + 13.33 \text{ mA} u(-t)$$

These two signals are plotted below as functions of time.



Example Problems

Capacitance and Equivalent Capacitance

1. The current flowing through a 25 nF capacitor is $i(t) = e^{-500t}u(t)$ mA. Derive equations for the voltage dropped over the capacitor and the instantaneous power consumed by the capacitor.
2. The voltage dropped over a 330 μF capacitor is $v(t) = 100t u(t)$ V. Derive equations for the current flowing through the capacitor and the instantaneous power consumed by the capacitor.
3. Calculate the equivalent capacitance of the circuit shown in figure 5.16.

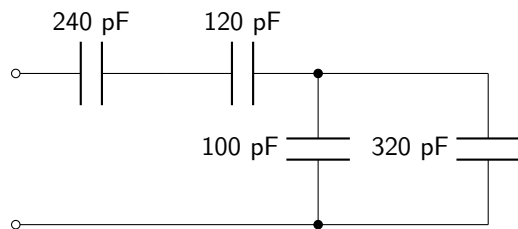


Figure 5.16: Circuit diagram for capacitance and equivalent capacitance question 3.

4. Calculate the equivalent capacitance of the circuit shown in figure 5.17.

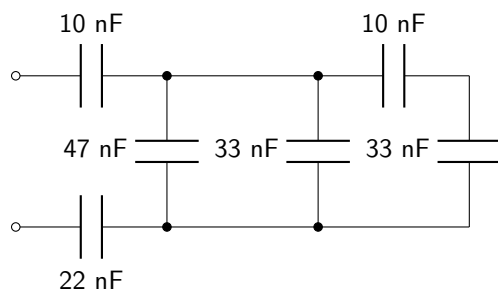


Figure 5.17: Circuit diagram for capacitance and equivalent capacitance question 4.

5. Determine the value of the capacitor C_X given that the circuit shown in figure 5.18 has an equivalent capacitance of 10 μF .

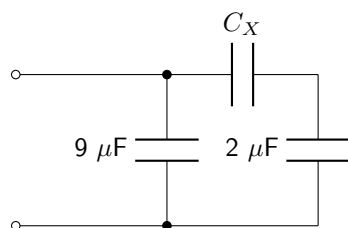


Figure 5.18: Circuit diagram for capacitance and equivalent capacitance question 5.

Resistor-Capacitor Circuits

6. Calculate an expression for $v(t)$ and $i(t)$ given the circuit shown in figure 5.19. The switch moves from position a to b at time of zero seconds.

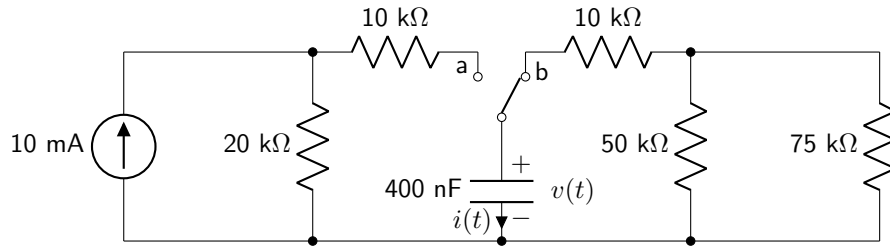


Figure 5.19: Circuit diagram for resistor-capacitor circuits question 6.

7. Calculate an expression for $v(t)$ and $i(t)$ given the circuit shown in figure 5.20. The switch opens at a time of zero seconds.

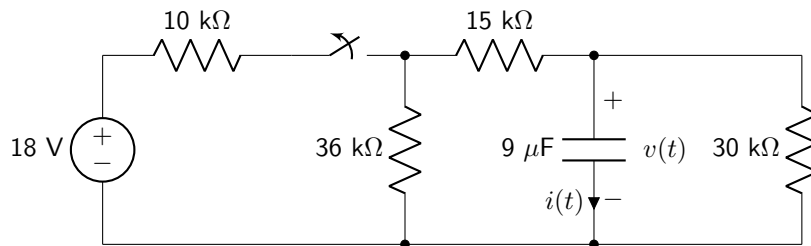


Figure 5.20: Circuit diagram for resistor-capacitor circuits question 7.

8. Calculate an expression for $v(t)$ and $i(t)$ given the circuit shown in figure 5.21. The switch closes at a time of zero seconds.

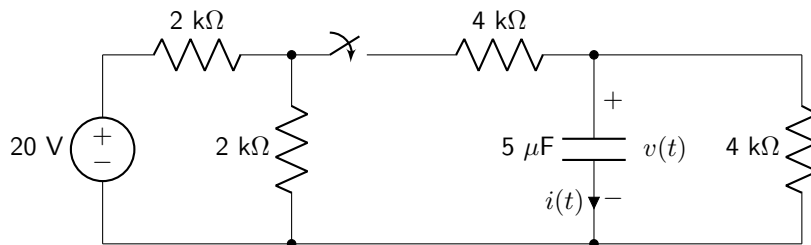


Figure 5.21: Circuit diagram for resistor-capacitor circuits question 8.

9. Calculate an expression for $v(t)$ and $i(t)$ given the circuit shown in figure 5.22. The switch closes at a time of zero seconds.

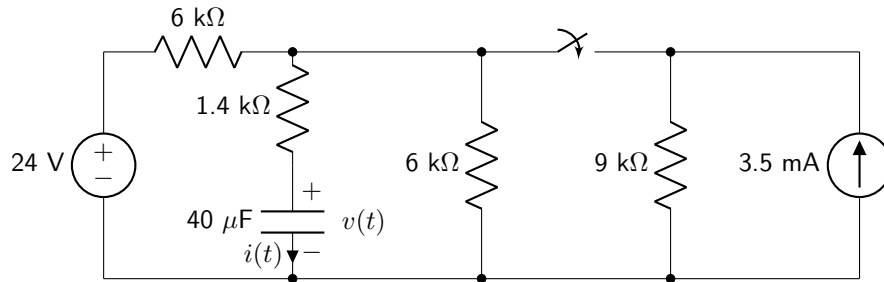


Figure 5.22: Circuit diagram for resistor-capacitor circuits question 9.

10. Calculate an expression for $v(t)$ and $i(t)$ given the circuit shown in figure 5.23. The switch opens at a time of zero seconds.

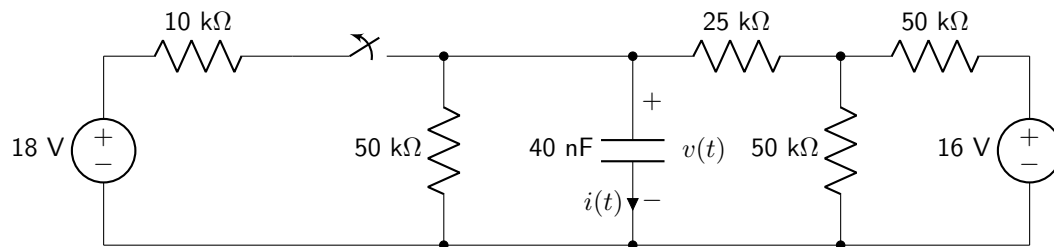


Figure 5.23: Circuit diagram for resistor-capacitor circuits question 10.

Practical Application: The 555 Timer

Now that comparators and capacitors are well-understood, the 555 timer can be analyzed. The 555 timer is an integrated circuit chip that is capable of performing many functions. Two of those functions are as an astable oscillator (clock) and a monostable pulse generator. The pinout diagram of a 555 timer integrated circuit chip is shown in figure PA.1. A pinout diagram is used to determine which pins on a chip correspond to each of the input and output signals of a functioning device. Each of these control signals will be explained as the functionality of the 555 timer is outlined below.

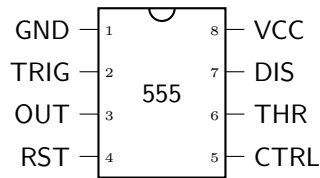


Figure PA.1: Pinout diagram of a 555 timer integrated circuit chip.

A 555 timer is composed of transistors, resistors, and other components. However the nature of each discrete component will be ignored and the device will be analyzed functionally. Functionally, the 555 timer consists of a voltage divider, two comparators, a digital device called an SR latch, a transistor, and an output driver. A schematic representation of this functional diagram is shown in figure PA.2. Each of the labeled connections corresponds to a pin on the integrated circuit chip (missing from the labeled connections is GROUND). Note the use of the term VCC which corresponds to the logical HIGH voltage value (usually 5 V). Each of the comparator power connections are not shown, but are VCC for the positive supply and ground for the negative supply.

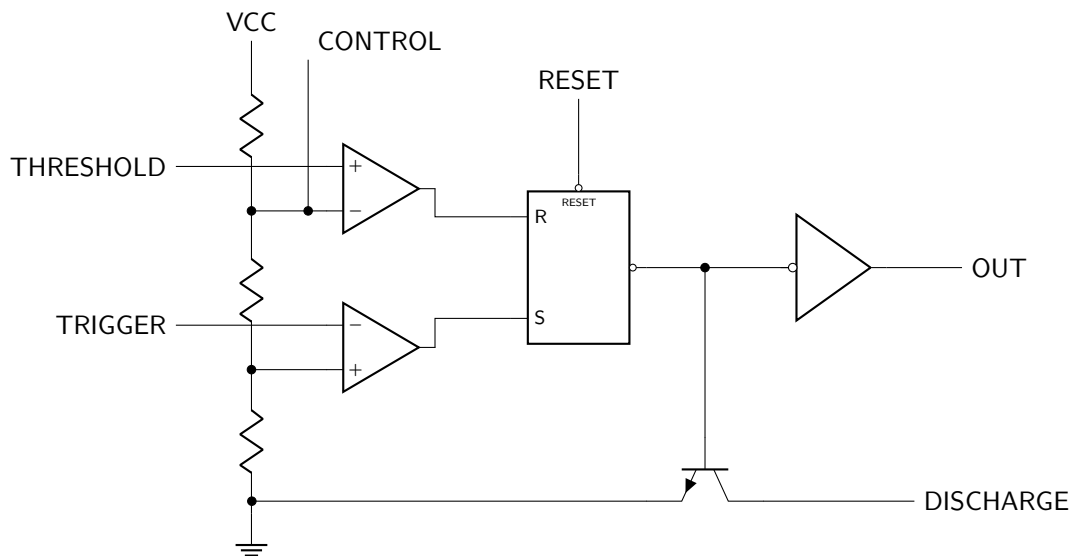


Figure PA.2: Functional schematic of a 555 timer.

On the left-hand side of the schematic are three resistors acting as a voltage divider. The reference

voltage for the upper comparator (the threshold comparator) is therefore $2/3 V_{CC}$ (which would be 3.3 V given a V_{CC} of 5 V). As soon as the voltage on the THRESHOLD pin exceeds 3.3 V, the output of the threshold comparator will saturate at V_{CC} . The threshold comparator output is connected to the R input of an SR latch. The R stands for reset. That will cause the output of the latch to be LOW; however, since the latch output is inverted (which is the meaning of the bubble), the latch output will be HIGH. Two things happen with this signal. First, the transistor will be activated, causing a short between the DISCHARGE pin and ground. Second, the output driver will invert the HIGH value and cause the output to be LOW (0 V). If the THRESHOLD voltage is less than 3.3 V, then the output of the comparator will saturate LOW. The reset input of the latch will not be activated.

The reference voltage for the lower comparator (the trigger comparator) is $1/3 V_{CC}$ (1.7 V given a V_{CC} of 5 V). If the voltage on the TRIGGER pin becomes less than 1.7 V, the output of the trigger comparator will saturate HIGH, causing the S input to be activated. This input stands for set. This will cause the output of the latch to be HIGH; however, since the latch output is inverted, the latch output will be LOW. This will cause the transistor to be deactivated, electrically disconnecting the DISCHARGE pin from ground. Second, the output driver will invert the LOW value and cause the output to be HIGH (5 V). If the TRIGGER voltage is greater than 1.7 V, then the output of the comparator will saturate LOW. The set input of the latch will not be activated.

If both the set and reset inputs (S and R) on the SR latch are LOW, the output of the latch will continue to be whatever value was previously stored on that output. (For more information about latches, please refer to my [Digital Systems textbook](#).)

The other two input pins are CONTROL and RESET. The CONTROL pin can be used to change the value of the voltage divider at the threshold comparator. It will not be discussed in this book. When unused, it should be connected to ground via a 10 nF capacitor that will remove any noise from the voltage divider. The RESET pin can be used to manually force the output of the timer to be LOW. When unused, it should be connected to V_{CC} .

Astable Mode

An astable oscillator is simply a name for a circuit whose output oscillates between two different values. This is particularly important for digital logic circuits that require clock signals. A clock signal is simply a square wave that oscillates between LOW and HIGH voltage values. An example of a clock signal is depicted in figure [PA.3](#).

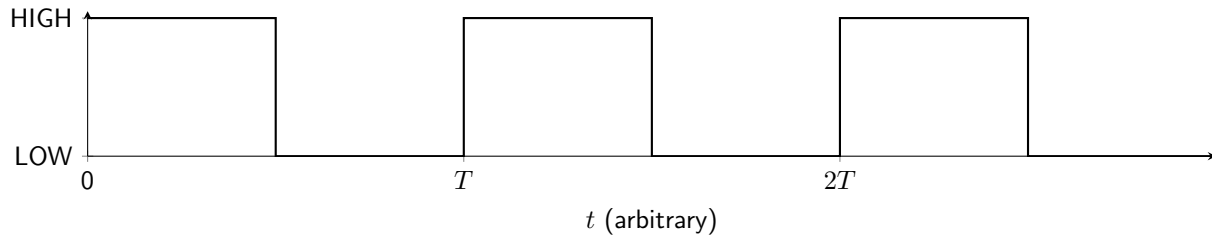


Figure PA.3: Graphical depiction of a digital clock signal.

The idea is to connect the TRIGGER and THRESHOLD pins and connect them to an RC circuit such that the voltage drop over the capacitor continuously changes. As it changes from $1/3 V_{CC}$ to $2/3 V_{CC}$, the output will oscillate between HIGH and LOW. The 555 timer in astable mode is connected as shown in figure PA.4. (The CONTROL pin is connected to a noise-reducing capacitor and the RESET pin is connected to VCC.)

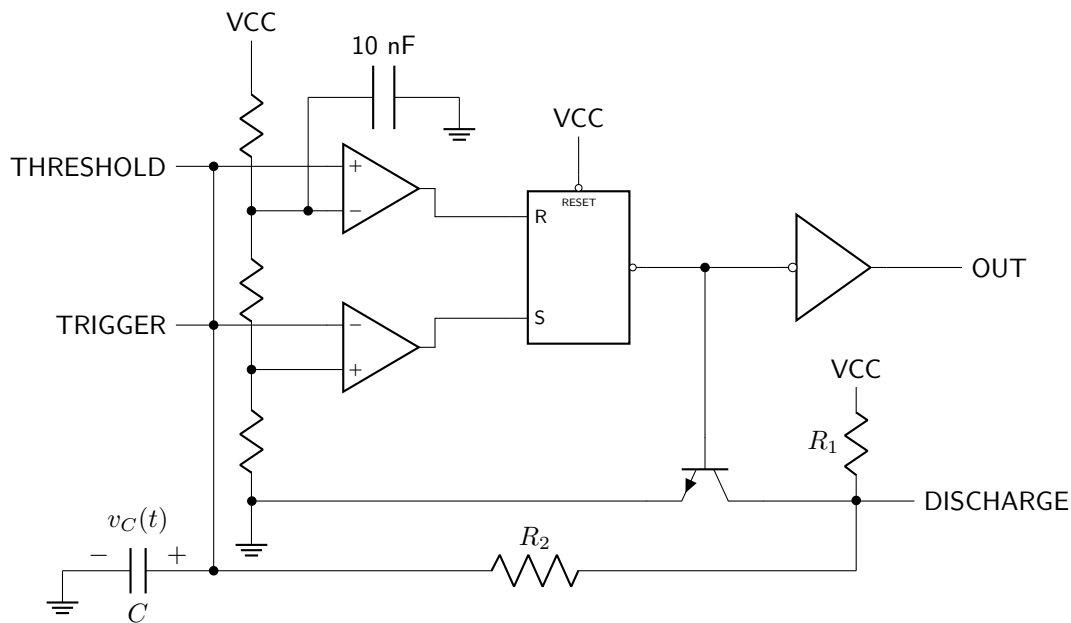


Figure PA.4: Functional schematic of a 555 timer connected in astable mode.

In the beginning, the capacitor will have no voltage drop. Because TRIGGER will be less than $1/3 V_{CC}$, the output will be HIGH, and the transistor will be deactivated. A pathway between VCC and ground exists through the capacitor, and it will begin to charge up. This is depicted in figure PA.5.

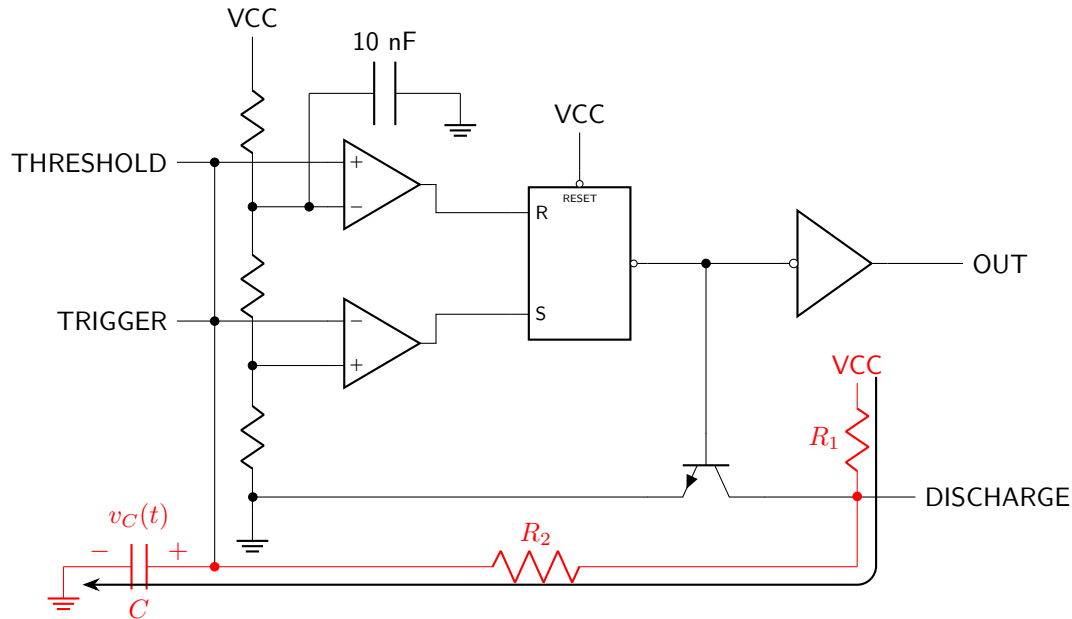


Figure PA.5: Functional schematic of a 555 timer connected in astable mode. When the output is HIGH (TRIGGER is less than $1/3$ VCC), the transistor is deactivated and current will flow as shown to charge the capacitor.

As soon as $v_C(t)$ reaches $1/3$ VCC, the trigger comparator will saturate LOW, disabling the set pin on the latch. Once $v_C(t)$ exceeds $2/3$ VCC, the threshold comparator will saturate HIGH, causing the latch to reset. This will activate the transistor and cause the capacitor to discharge through R_2 , as shown in figure PA.6.

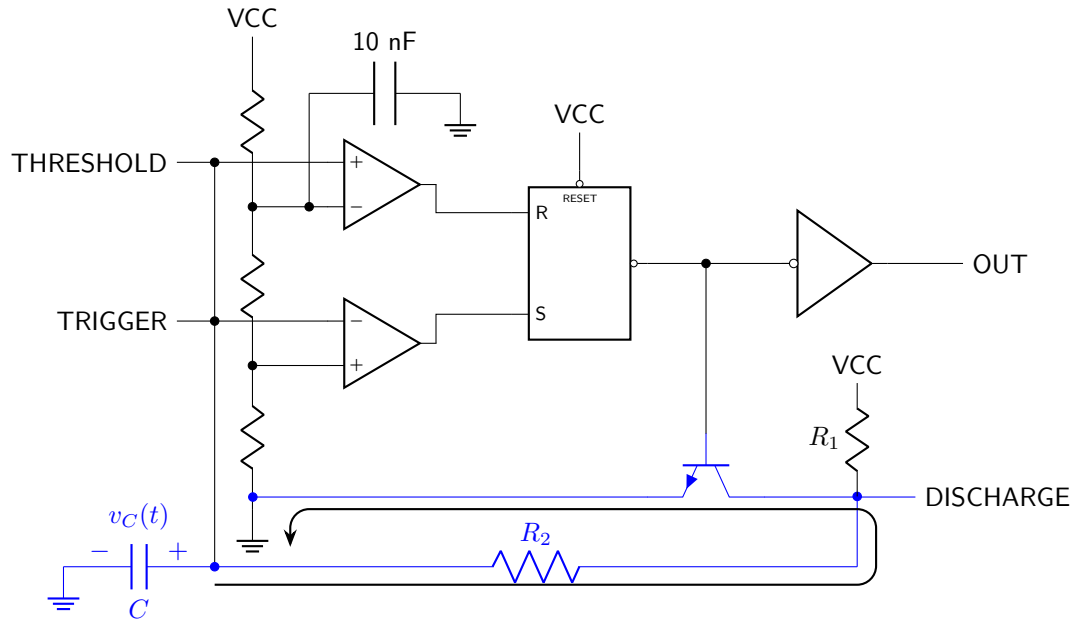


Figure PA.6: Functional schematic of a 555 timer connected in astable mode. When the output is LOW (THRESHOLD is greater than $2/3 V_{CC}$), the transistor is activated and current will flow as shown to discharge the capacitor.

Once the voltage drop over the capacitor decreases below $1/3 V_{CC}$, the trigger comparator will then saturate HIGH. This cycle will repeat itself where the capacitor charges from $1/3 V_{CC}$ to $2/3 V_{CC}$ and then discharges back to $1/3 V_{CC}$. Meanwhile, the digital nature of the comparators and the output driver creates a square wave output signal.

Because the capacitor charges through R_1 and R_2 and discharges only through R_2 , the capacitor will spend more time charging than discharging. The output therefore does not have an equal amount of time spent HIGH and LOW. (In other words, the duty cycle will be greater than 50%.) An example waveform that shows $v_C(t)$ as well as the output signal is shown in figure PA.7. In this example, $R_1 = R_2 = 1 \text{ k}\Omega$ and $C = 10 \mu\text{F}$. (Note also that the first waveform has an even higher duty cycle due to the fact that the capacitor must charge from 0 rather than from $1/3 V_{CC}$.)

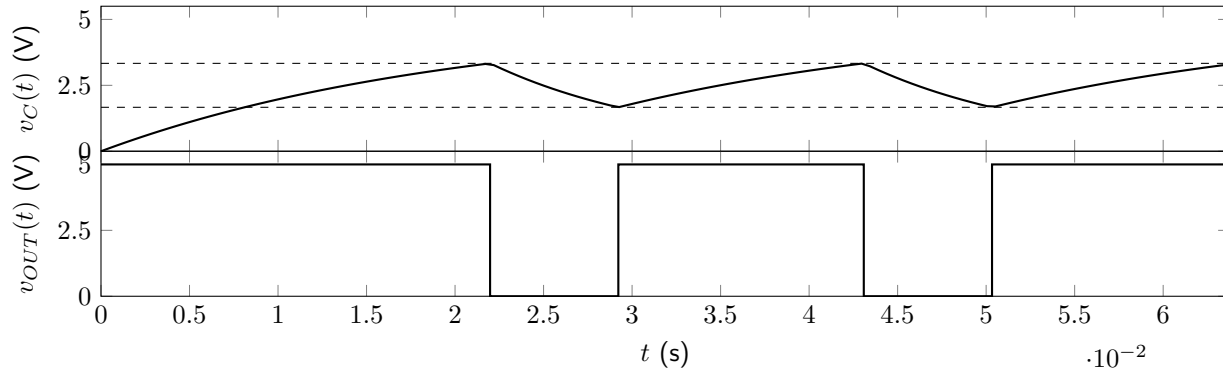


Figure PA.7: Voltage drop over the capacitor (top graph) and output voltage (bottom graph) in astable mode. $R_1 = R_2 = 1 \text{ k}\Omega$ and $C = 10 \text{ }\mu\text{F}$.

Equations 5.12 and 5.15 can be used to determine the amount of time the output signal is HIGH (T_{HIGH}) as well as the amount of time the output signal is LOW (T_{LOW}). These values are expressed in equations PA.1 and PA.2.

$$T_{HIGH} = C(R_1 + R_2) \ln 2 \quad (\text{PA.1})$$

$$T_{LOW} = CR_2 \ln 2 \quad (\text{PA.2})$$

Monostable Mode

In monostable mode, the 555 timer is capable of creating a single pulse upon a trigger event. The pulse-width will be related to the value of the resistor and capacitor used in the external circuitry. In this case, the TRIGGER pin will be connected to an external trigger (perhaps a pushbutton or another electronic signal). When the value on the TRIGGER pin is less than $1/3 \text{ VCC}$, it will cause the capacitor to charge up a single time, creating one pulse. The 555 timer connected in monostable mode is shown in figure PA.8.

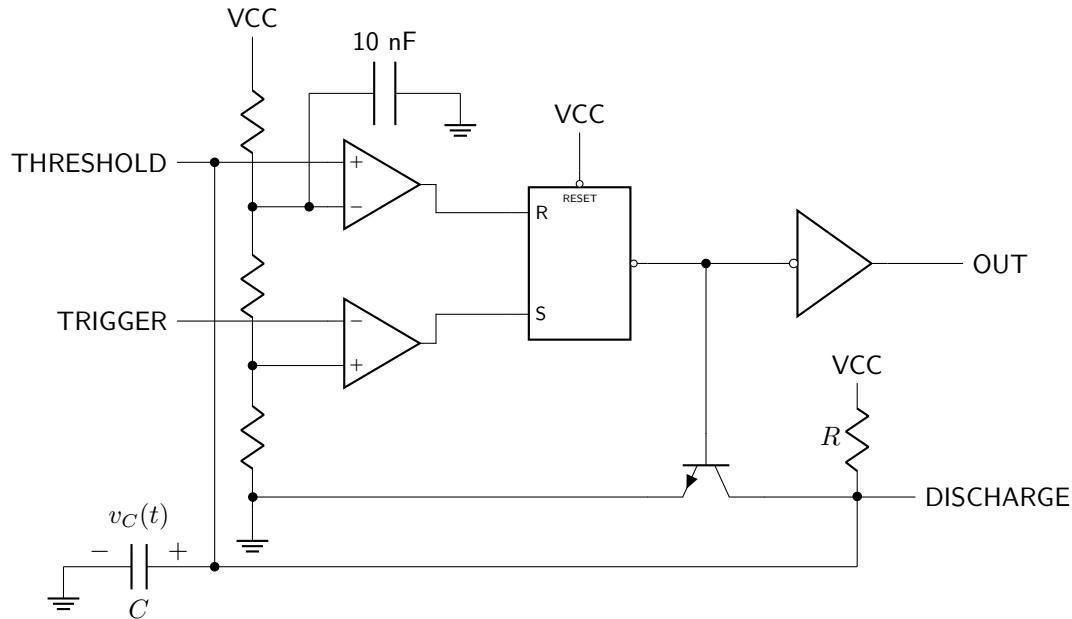


Figure PA.8: Functional schematic of a 555 timer connected in monostable mode.

In the beginning, v_C will be zero. The THRESHOLD value will be less than $2/3 V_{CC}$, and as long as TRIGGER is held at a voltage higher than $1/3 V_{CC}$, the OUTPUT will be LOW and the capacitor will be connected to ground via the transistor. This will continue to be the case as long as the TRIGGER input does not change. As soon as a voltage less than $1/3 V_{CC}$ is presented on the TRIGGER pin, the OUTPUT signal will become HIGH. This will cause current to flow and the capacitor to charge, as shown in figure PA.9.

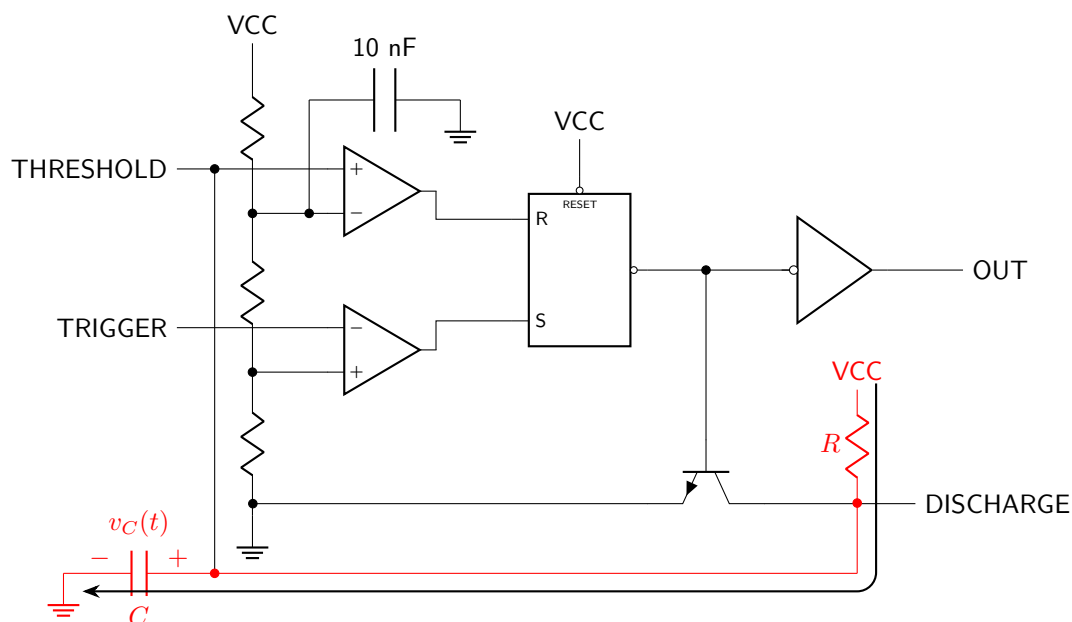


Figure PA.9: Functional schematic of a 555 timer connected in monostable mode. When a voltage of less than $1/3 V_{CC}$ is present on TRIGGER, the capacitor will charge, as shown.

Because the TRIGGER pulse was short-lived, the trigger comparator will saturate LOW as soon as the TRIGGER signal returns HIGH. This will cause the SR latch output to remain unchanged. As soon as the voltage drop over the capacitor reaches $2/3 V_{CC}$, the threshold comparator will saturate HIGH, causing the SR latch to reset (OUTPUT will be LOW gain). The capacitor will discharge directly to ground through the transistor without going through any resistance, causing an abrupt discharge to zero. The output will remain in equilibrium unless the TRIGGER pin is sent LOW again.

A graph of the TRIGGER pulse and the resulting values of $v_C(t)$ and the OUTPUT signal are shown in figure PA.10.

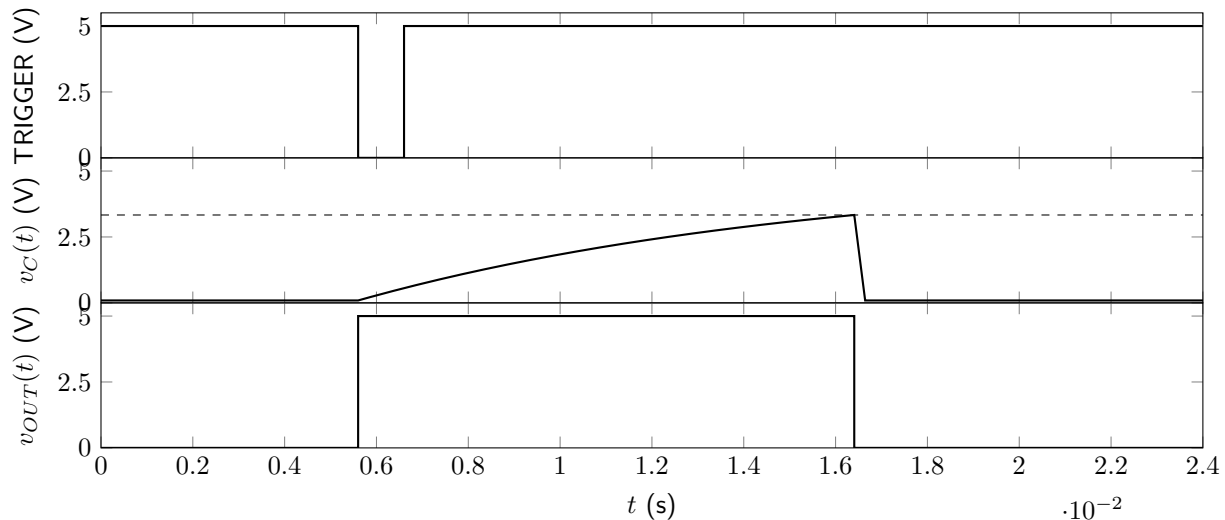


Figure PA.10: Trigger input (top graph), voltage drop over the capacitor (center graph), and output voltage (bottom graph) in monostable mode. $R = 1 \text{ k}\Omega$ and $C = 10 \text{ }\mu\text{F}$.

Equation 5.15 can be used to determine the amount of time the output signal is HIGH (T_{HIGH}). This value is expressed in equation PA.3.

$$T_{HIGH} = CR \ln 3 \quad (\text{PA.3})$$

Because the TRIGGER pin is connected externally, there are two important consequences. First, the output will remain HIGH for the entire time that the TRIGGER pin voltage is lower than $1/3 V_{CC}$, which may be longer than the amount of time described in equation PA.3. Therefore it is important to ensure that the TRIGGER pulse-width is shorter than the amount of time given in equation PA.3.

Second, any subsequent TRIGGER event needs to occur after the capacitor has charged to $2/3 V_{CC}$ and then discharged back to zero again. Otherwise, the second event will not generate an output pulse.

6 Inductors and RL Circuits

Inductors are circuit elements that are capable of storing and releasing energy in the form of a magnetic field. Most inductors are comprised of a coil of wire, possibly coiled around a ferrite core. (Even a straight wire can create a magnetic field, but the effect is very small when compared to coiled wire inductors.) As current flows through the inductor, a magnetic field is generated.

The property of an inductor that relates to its ability to store magnetic field energy is known as inductance (L) which is measured in units of H (henrys). The construction of inductors takes on many forms, and the equation for inductance in a device depends on the construction and geometry of that particular inductor. For a cylindrical coil of wire, the inductance depends on the permeability of the core material (μ), the number of turns of wire (N), the area of the coil (A), and the length of the coil (l), as shown in equation 6.1.

$$L = \frac{\mu N^2 A}{l} \quad (6.1)$$

Many geometries can be used for inductors, and they can contain an air core or a ferrite core depending on how high the inductance requirements are. Straight coils of wire (known as solenoids) are relatively simple to construct but are not as efficient as toroidal designs that are able to keep more magnetic flux isolated inside of the inductor.

The permeability of a material refers to how much an object magnetizes in response to an applied magnetic field. It is equal to the permeability of free space multiplied by the relative permeability of a particular material, defined by equation 6.2.

$$\mu = \mu_0 \mu_r \quad (6.2)$$

The relative permeability of materials can be looked up, but is equal to one for free space and approximately 200,000 for ferromagnetic materials such as iron. The permeability of free space is given in equation 6.3.

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m} \quad (6.3)$$

The circuit symbol for an inductor is shown in figure 6.1. Just as the capacitor symbol looks like parallel plates, the inductor symbol resembles a coil of wire.



Figure 6.1: The circuit symbol for an inductor.

Variable inductors are not common but do exist. One type of variable inductor construction is to have a straight coil of wire with a moveable ferrite core. As the ferrite core is moved into the coil, the inductance increases. As the ferrite core is moved out of the coil, the inductance decreases.

6.1 Parasitic Effects

Inductors are made of coils of wire. To create a high inductance device, a lot of wire is required. This generates parasitic side-effects. Wire contains resistance which, while it can be safely ignored when small amounts are used, can no longer be ignored when large amounts are used. To create physically small inductors, wire with a smaller surface area (high gauge) are used, which increases the resistance of the wire. The resistance of an inductor is known as *parasitic resistance*. In addition, there can be *parasitic capacitance* in an inductor due to the presence of charge in many parallel wires. These parasitic effects may vary with the frequency of the applied signal. A model of an inductor, including parasitic resistance and capacitance, is shown in figure 6.2.

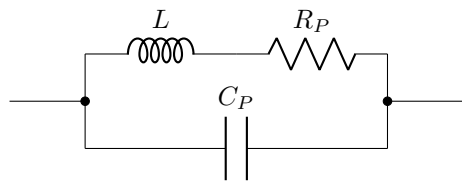


Figure 6.2: An inductor model including parasitic resistance (R_P) and parasitic capacitance (C_P).

This textbook will ignore the presence of parasitic effects for the most part. However, it is important to understand that the parasitic effects of an inductor are non-trivial in real-world applications. Some sections in this textbook will therefore consider examples that include parasitic resistance (while parasitic capacitance will be ignored).

6.2 Measuring Inductance

Inductors generally cannot be measured with a digital multimeter, as the inclusion of the necessary components to measure inductance would increase the cost prohibitively. They can, however, be measured with LRC meters. Much like with a capacitor, an LRC meter applies an AC voltage, measures the AC current, and uses the phase and amplitude differences to calculate the inductance.

Inductance must be measured from one end of the inductor (or equivalent inductance) to the other, in parallel, as shown in figure 6.3.

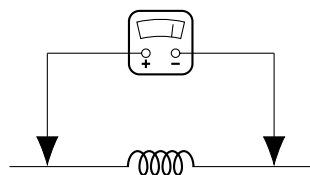


Figure 6.3: A diagram of how inductance is measured in parallel (across a component or components).

6.3 Voltage-Current Relationship in an Inductor

The relationship between voltage and current in an inductor can be derived using Faraday's law of induction. The full derivation will not be shown in this text. The voltage drop over an inductor is given by equation 6.4.

$$v(t) = L \frac{di(t)}{dt} \quad (6.4)$$

Based equation 6.4, if the current remains unchanging through time, there will be no voltage drop over an inductor. Therefore, in DC conditions, an inductor can be replaced by a short circuit. This analysis technique will be used frequently in this book to explore the transient response of resistor-inductor circuits.

The current flowing through an inductor can be found by integrating both sides of equation 6.4. The result is given in equation 6.5.

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(\tau) d\tau \quad (6.5)$$

Although calculus is required to fully analyze the relationship between voltage and current in an inductor, the relationship between the two variables is linear. This means that inductors are linear circuit elements, and all of the techniques discussed in this book (such as superposition and Thévenin equivalence, for example) are still relevant with inductive circuits.

6.4 Equivalent Inductance

Just as it was possible to combine resistors together to find an equivalent resistance, and capacitors to find an equivalent capacitance, it is possible to combine inductors together to find an equivalent inductance. The rules (and mathematical relationships) for calculating series and parallel combinations of inductors are explained below.

6.4.1 Series Combinations of Inductors

Inductors in series all share a common branch, as shown in figure 6.4. Each individual voltage drop is shown (numbered with subscripts), as well as the total voltage drop over the combination of inductors ($v(t)$). The current flow through all of the inductors is the same and is labeled $i(t)$.

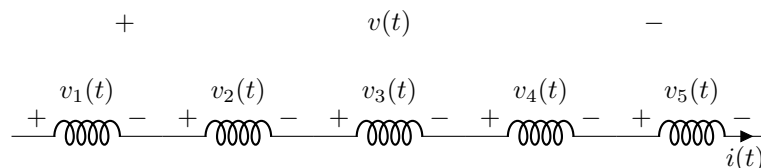


Figure 6.4: Five inductors in series in a circuit.

As $v(t)$ is equal to the sum of each individual voltage drop, the current-voltage relationship in inductors

(equation 6.4) can be used to determine the equivalent inductance of series combinations of inductors. This is shown in equation 6.6 and derived below.

$$L_{EQ} = L_1 + L_2 + L_3 + \dots + L_n \quad (6.6)$$

DERIVATION

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) + v_3(t) + \dots + v_n(t) \\ L_{EQ} \frac{di(t)}{dt} &= L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + L_3 \frac{di(t)}{dt} + \dots + L_n \frac{di(t)}{dt} \\ L_{EQ} &= L_1 + L_2 + L_3 + \dots + L_n \end{aligned}$$

Inductors in series therefore combine similarly to resistors in series. The equivalent inductance cannot be smaller than the largest inductor in the series combination.

6.4.2 Parallel Combinations of Inductors

Inductors in parallel all share two nodes, as shown in figure 6.5. Each individual current flow is shown (numbered with subscripts), as well as the total current flow through the combination of inductors ($i(t)$). The voltage drop over all of the inductors is the same and is labeled $v(t)$.

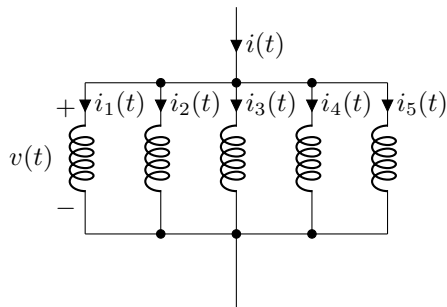


Figure 6.5: Five inductors in parallel in a circuit.

As $i(t)$ is equal to the sum of each individual current flow, the current flow through an inductor (equation 6.5) can be used to determine the equivalent inductance of parallel combinations of inductors. This is shown in equation 6.7 and derived below.

$$\frac{1}{L_{EQ}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \dots + \frac{1}{L_n} \quad (6.7)$$

DERIVATION

$$i(t) = i_1(t) + i_2(t) + i_3(t) + \dots + i_n(t)$$

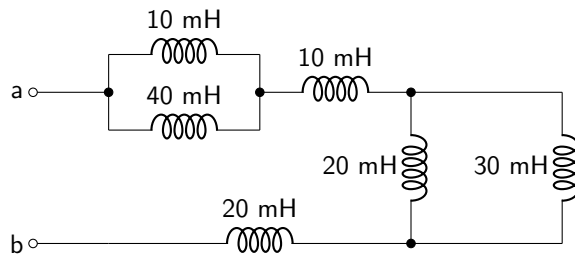
$$\frac{1}{L_{EQ}} \int_{-\infty}^t v(\tau) d\tau = \frac{1}{L_1} \int_{-\infty}^t v(\tau) d\tau + \frac{1}{L_2} \int_{-\infty}^t v(\tau) d\tau + \frac{1}{L_3} \int_{-\infty}^t v(\tau) d\tau + \dots + \frac{1}{L_n} \int_{-\infty}^t v(\tau) d\tau$$

$$\frac{1}{L_{EQ}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \dots + \frac{1}{L_n}$$

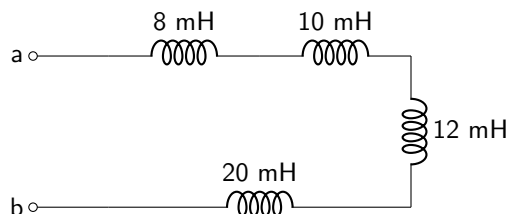
Inductors in parallel therefore combine similarly to resistors in parallel. The equivalent inductance cannot be larger than the smallest inductor in the parallel combination.

Example: Calculating equivalent inductance

Calculate the equivalent inductance between nodes a and b.



Combine both parallel combinations of inductors. The 10 mH and 40 mH pair have an equivalent inductance of 8 mH. The 20 mH and 30 mH pair have an equivalent inductance of 12 mH. The circuit can now be re-drawn.



The equivalent inductance can be calculated by summing together the four inductances. L_{EQ} is equal to 50 mH.

6.5 Resistor-Inductor (RL) Circuits

Resistor-inductor (RL) circuits are circuits that contain one inductor (or combinations of inductors that can be reduced down to one equivalent inductance) as well as at least one resistive element. In this chapter, these circuits will be analyzed given an abrupt change from one set of DC conditions to another. The short-term difference in output that occurs as a result of this change in DC conditions is known as a transient analysis.

An important property of inductors that will aid in the analysis of the transient response is that **the current through an inductor cannot change instantaneously**. That is: $i(0^-) = i(0^+)$. In an ideal analysis where inductors have no parasitic capacitance (which is what this textbook will assume), the voltage dropped over an inductor can change instantaneously and therefore be discontinuous in time.

6.5.1 Discharging Transient Response

The discharging response is a special case of the transient response that specifically looks at what happens when an inductor is initially connected to a DC current that is suddenly removed from a circuit. In other words: the inductor, after being disconnected from the power supply, discharges its stored energy through the resistive element(s) of the circuit. The circuit diagram shown in figure 6.6 will be used to analyze this discharging response. The switch is in position a for a long time before suddenly moving to position b at time $t = 0$.

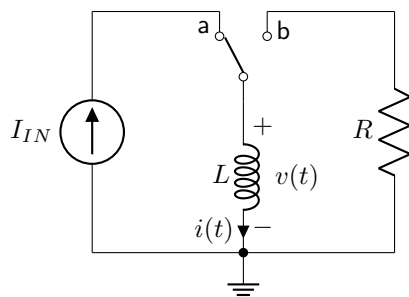


Figure 6.6: An RL circuit used to analyze the discharging transient response.

Kirchhoff's laws can be used to analyze the circuit in the initial condition. Because the only two elements are the current source and the inductor, and they are in series, the initial current flow through the inductor must be equal to the source current. In other words: $i(0) = I_{IN}$.

Kirchhoff's laws are also used to analyze the circuit in the discharging condition, once the switch has been flipped to position b. The voltage drop over the inductor ($v(t)$, defined in equation 6.4) is equal to the voltage drop over the resistor. Because the two components are in series they share a current flow of $i(t)$. The discharging transient response is derived from these properties in equation 6.8 and derived below.

$$0 = \frac{di(t)}{dt} + \frac{R}{L}i(t) \quad (6.8)$$

DERIVATION

$$\begin{aligned}
 0 &= \frac{v(t)}{R} + i(t) \\
 &= \frac{L}{R} \frac{di(t)}{dt} + i(t) \\
 &= \frac{di(t)}{dt} + \frac{R}{L} i(t)
 \end{aligned}$$

The equation for the current flow in this transient discharging analysis is a first order, linear, ordinary differential equation. This is consistent with the fact that the circuit itself is a first order, linear circuit! The solution to the differential equation will not be derived in this book but is given in equation 6.9. Note the similarities with the RC transient equation.

$$i(t) = i(0^+)e^{-\frac{t}{L/R}}u(t) + i(0^-)u(-t) \quad (6.9)$$

Using equation 6.4, the transient response of the voltage drop over the discharging circuit can also be determined. This is shown in equation 6.10. Note that initial voltage drop over an inductor will be zero, as an inductor acts like a short in DC steady-state conditions.

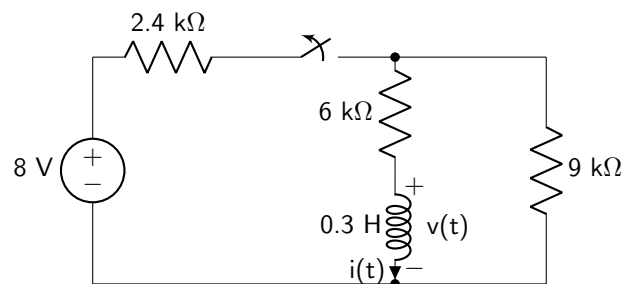
$$v(t) = -Ri(0^+)e^{-\frac{t}{L/R}}u(t) \quad (6.10)$$

The quantity L/R defines the quickness that the circuit discharges to zero and is known as the *RL time constant*. The mathematical symbol for the RL time constant is τ . Specifically, τ defines the amount of time it takes the current flow to decay to 36.8% of its maximum value.

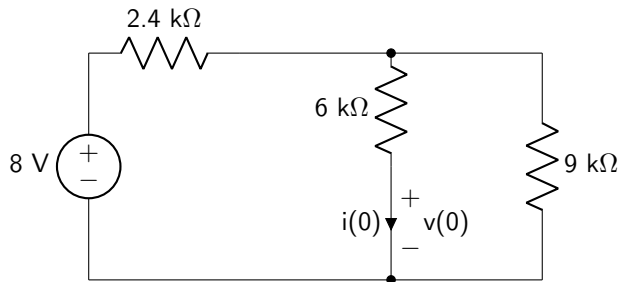
If there is more than one resistive element in the discharging circuit, then the value of R in the time constant is equal to the equivalent resistance as seen by the inductor.

Example: Discharging transient response

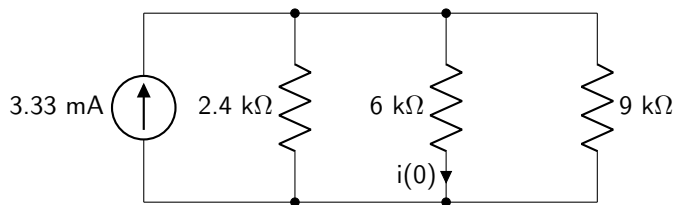
Calculate $i(t)$ and $v(t)$ for the circuit below. The switch opens at a time of 0 seconds.



The initial conditions can be calculated. Re-draw the circuit to represent the time before the switch opens. Because the circuit is under DC conditions, the inductor can be represented as a short.



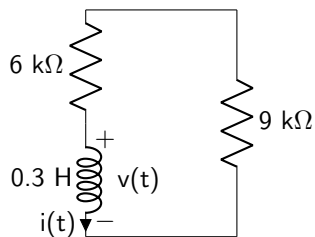
The initial voltage drop will be zero. Source transformation can be used to calculate the initial current.



Use a current divider to calculate $i(0)$.

$$\begin{aligned} i(0^-) &= 3.33 \text{ mA} \left(\frac{1.44 \text{ k}\Omega}{6 \text{ k}\Omega} \right) \\ &= 0.8 \text{ mA} \end{aligned}$$

Analyze the discharging circuit. Re-draw for time $t \geq 0$.

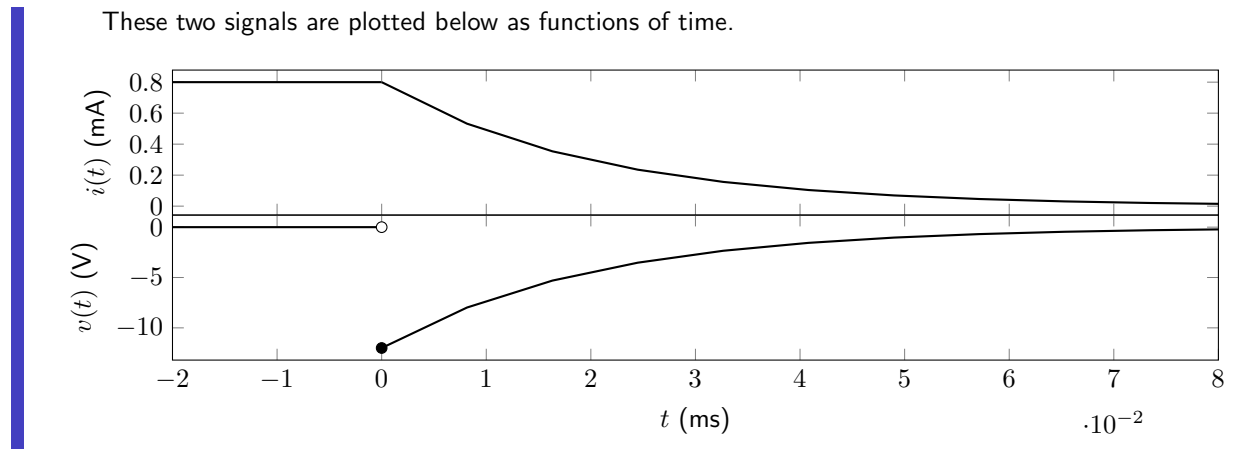


The equivalent resistance seen by the inductor is 15 kΩ. This can be used to calculate the RL time constant.

$$\begin{aligned} \tau &= \frac{0.3 \text{ H}}{15 \text{ k}\Omega} \\ &= 2 \times 10^{-5} \text{ s} \end{aligned}$$

The equations for $i(t)$ and $v(t)$ can now be derived using equations 6.9 and 6.10.

$$\begin{aligned} i(t) &= 0.8 \text{ mA} e^{-50000t} u(t) + 0.8 \text{ mA} u(-t) \\ v(t) &= -12 \text{ V} e^{-50000t} u(t) \end{aligned}$$



6.5.2 Charging Transient (Step) Response

The charging transient (step) response of a circuit corresponds to how the voltage and/or current changes with respect to a sudden application of current to the circuit components. This is another specific case of the general RL transient response, which will be discussed in section 6.5.3. Specifically, the charging transient response refers to a fully discharged inductor suddenly being connected to a circuit containing at least one resistive element and a power supply. The circuit diagram shown in figure 6.7 will be used to analyze the charging response. The switch is in position b for a long time before suddenly moving to position a at time $t = 0$.

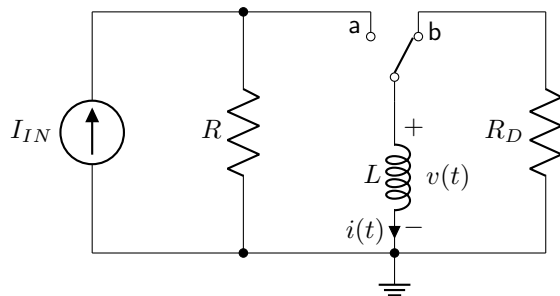


Figure 6.7: An RL circuit used to analyze the charging transient (step) response.

In the initial state of the circuit, all of the energy that may have been stored in the inductor has completely discharged through the discharging resistor (R_D) giving an initial current flow of 0 mA.

In the charging condition (when the switch has been moved to position a), Kirchhoff's current law can be used to analyze the current flow through each circuit element. From this, the differential equation for $i(t)$ can be derived. This solution is shown in equation 6.11, and the derivation is given below. The term $i(\infty)$ corresponds to the final, long-term value of the current flow through the inductor. It will be equal to I_{IN} in

the case of the circuit shown in figure 6.7.

$$0 = \frac{di(t)}{dt} + \frac{i(t) - i(\infty)}{L/R} \quad (6.11)$$

DERIVATION

$$\begin{aligned} 0 &= -i(\infty) + \frac{v(t)}{R} + i(t) \\ &= -i(\infty) + \frac{L}{R} \frac{di(t)}{dt} + i(t) \\ &= \frac{di(t)}{dt} + \frac{i(t) - i(\infty)}{L/R} \end{aligned}$$

The solution to this differential equation (assuming that the initial current flow through the inductor is 0 mA) is given in equation 6.12. Again, note the similarities with the RC circuit step response.

$$i(t) = i(\infty) \left[1 - e^{-\frac{t}{L/R}} \right] u(t) \quad (6.12)$$

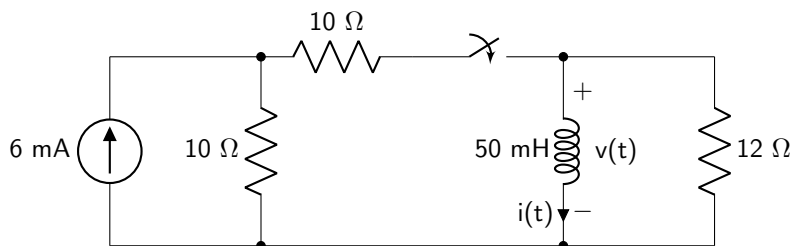
The value of the RL time constant, τ , is still equal to the equivalent inductance divided by the equivalent resistance as seen by the inductor. The time constant in a charging circuit dictates how much time it takes the circuit to charge up to 63.2% of the maximum current value.

Equation 6.4 can be used to find the voltage drop over the inductor during the transient charging process. The result is shown in equation 6.13.

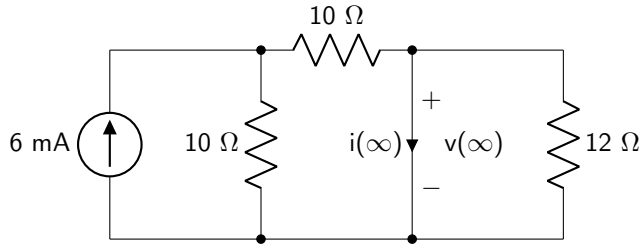
$$v(t) = Ri(\infty)e^{-\frac{t}{L/R}} u(t) \quad (6.13)$$

Example: Charging transient response

Calculate $i(t)$ and $v(t)$ for the circuit below. The switch closes at a time of 0 seconds.



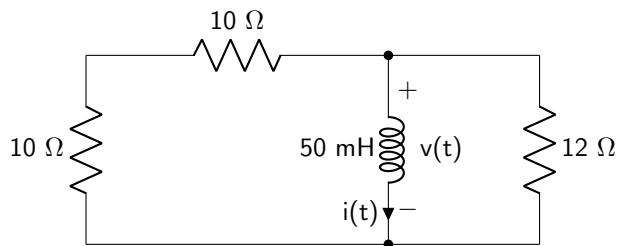
The final (steady-state) conditions can be calculated. Re-draw the circuit to represent the time well after the switch closes. Because the circuit is under DC conditions, the inductor can be represented as a short. The final voltage is 0 V.



Use a current divider to calculate $i(\infty)$.

$$\begin{aligned} i(\infty) &= 6 \text{ mA} \left(\frac{5 \Omega}{10 \Omega} \right) \\ &= 3 \text{ mA} \end{aligned}$$

Analyze the charging circuit. Re-draw for time $t \geq 0$. Deactivate the current source by replacing it with an open to calculate the equivalent resistance as seen by the inductor.



The equivalent resistance is calculated below.

$$\begin{aligned} R_{EQ} &= (10 \Omega + 10 \Omega) // 12 \Omega \\ &= 20 \Omega // 12 \Omega \\ &= 7.5 \Omega \end{aligned}$$

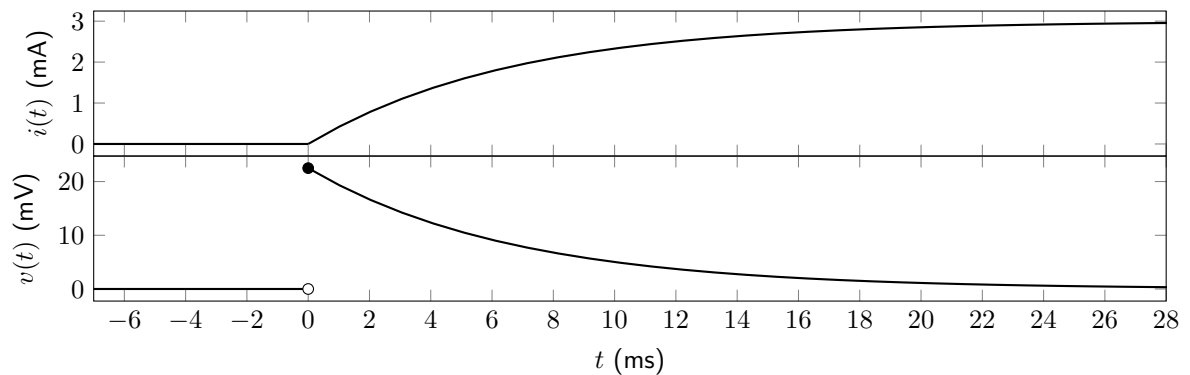
Calculate the RL time constant.

$$\begin{aligned} \tau &= \frac{0.05 \text{ H}}{7.5 \Omega} \\ &= 0.0067 \text{ s} \end{aligned}$$

The equations for $i(t)$ and $v(t)$ can now be derived using equations 6.12 and 6.13.

$$\begin{aligned} i(t) &= 3 \text{ mA} [1 - e^{-150t}] u(t) \\ v(t) &= 22.5 \text{ mV } e^{-150t} u(t) \end{aligned}$$

These two signals are plotted below as functions of time.



6.5.3 General Transient Response

Two special cases of the RL transient response have been analyzed: the discharging response ($i(\infty) = 0$) and the step response ($i(0) = 0$). In general, the initial current flow through an inductor and the final value of the current flow through the inductor can be any arbitrary values. The most general form of the equation for the transient response of an RL circuit is therefore given in equation 6.14.

$$i(t) = \left[i(\infty) + (i(0^+) - i(\infty)) e^{-\frac{t}{L/R}} \right] u(t) + i(0^-) u(-t) \quad (6.14)$$

The voltage drop over the inductor can be calculated using equation 6.4. This is shown in equation 6.15.

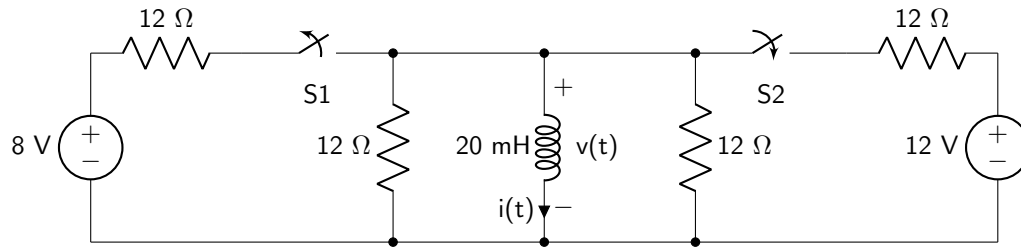
$$v(t) = R [i(\infty) - i(0^+)] e^{-\frac{t}{L/R}} u(t) \quad (6.15)$$

In the general transient response, the RL time constant refers to how much time it takes for the current flow through the inductor to reach a value given by equation 6.16.

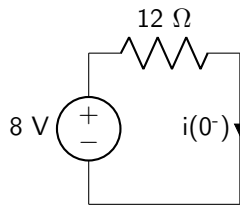
$$i(\tau) = 0.632i(\infty) + 0.368i(0^+) \quad (6.16)$$

Example: General transient response

Calculate $i(t)$ and $v(t)$ for the circuit below. Switch S1 opens at a time of 0 seconds and switch S2 closes at a time of 0.005 seconds.

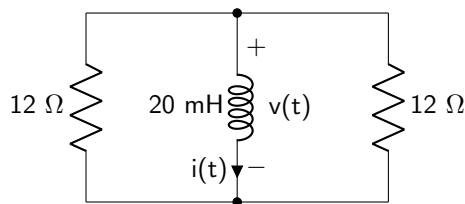


Analyze the initial conditions of the circuit. Because the inductor acts as a short in the DC conditions, the initial value of the voltage $v(0^-)$ is zero.



Using Ohm's law, the initial current flow through the inductor is 666.67 mA.

The RL time constant that dictates the discharge rate can be calculated by looking at the circuit with both switches open.



The equivalent resistance seen by the inductor is 6 Ω. The RL time constant is therefore 0.003 s. For the first 5 ms, the circuit will be acting as a discharging circuit with an equation given below.

$$i(t) = 666.67 \text{ mA } e^{-300t} [u(t) - u(t - 0.005)] + 666.67 \text{ mA } u(-t)$$

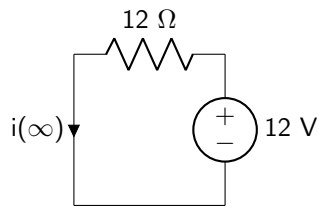
The voltage during this time period can be found by using equation 6.15.

$$v(t) = -4 \text{ V } e^{-300t} [u(t) - u(t - 0.005)]$$

To analyze the situation when switch S2 closes, the current at 5 ms should be calculated using the equation above. This will act as the initial condition for this time period.

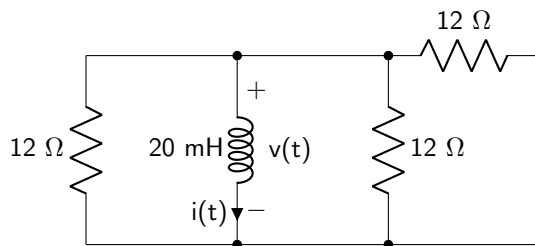
$$i(0.005) = 666.67 e^{-(300)(0.005)} \text{ mA} = 148.75 \text{ mA}$$

The final conditions should be analyzed to calculate $i(\infty)$. The inductor can be replaced by a short in steady-state DC conditions. The steady-state value of the voltage will be zero.



The steady-state current flow through the inductor is 1000 mA.

The time period after 5 ms should be analyzed to find the equivalent resistance seen by the inductor to calculate the new RL time constant. The voltage source has been deactivated.



The equivalent resistance as seen by the inductor is now 4Ω . The new RL time constant is 0.005 s . For the remaining period of time after switch S2 closes, the circuit will act as a charging circuit with an equation given below.

$$i(t) = \left[1000 \text{ mA} - 851.25 \text{ mA} e^{-200(t-0.005)} \right] u(t - 0.005)$$

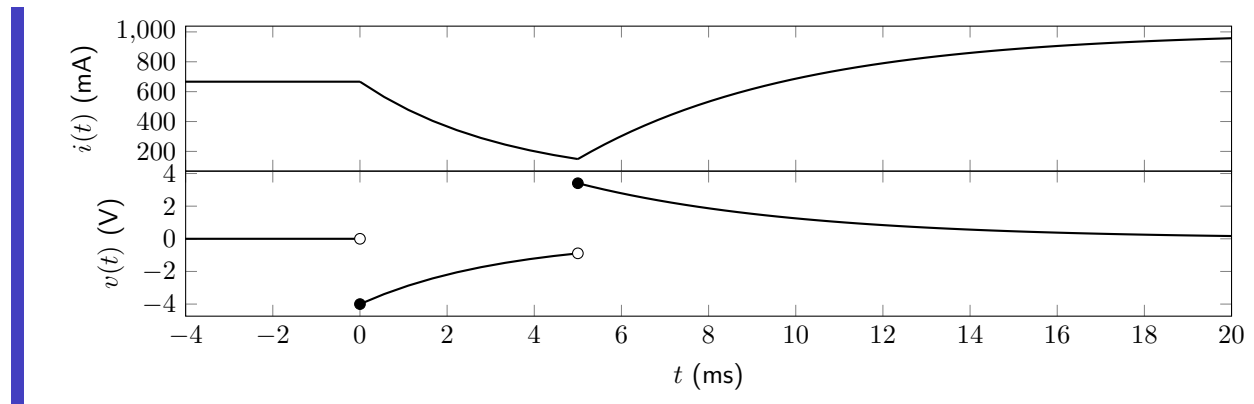
The voltage during this time period can be found by using equation 6.15.

$$v(t) = 3.405 \text{ V} e^{-200(t-0.005)} u(t - 0.005)$$

The full equations for current and voltage can be found by summing the two different signals together.

$$\begin{aligned} i(t) &= 666.67 \text{ mA} e^{-300t} [u(t) - u(t - 0.005)] \\ &\quad + \left[1000 \text{ mA} - 851.25 \text{ mA} e^{-200(t-0.005)} \right] u(t - 0.005) \\ &\quad + 666.67 \text{ mA} u(-t) \\ v(t) &= -4 \text{ V} e^{-300t} [u(t) - u(t - 0.005)] + 3.405 \text{ V} e^{-200(t-0.005)} u(t - 0.005) \end{aligned}$$

These two signals are plotted below as functions of time.



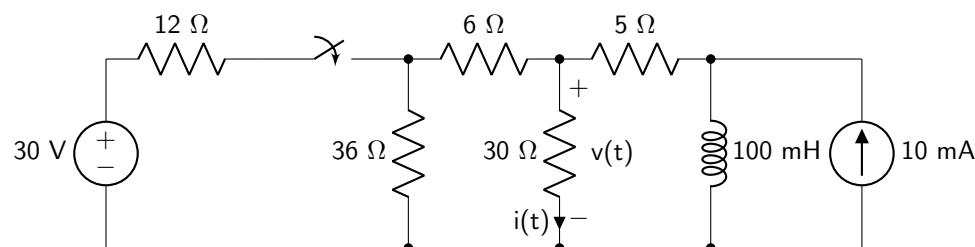
All of the equations in this section of the text have discussed finding the current flow through the inductor. It is also possible to derive a first order equation to find voltage over or current through any arbitrary component in a circuit. It is important to note that the initial conditions may be discontinuous. This is because only the current flowing through an inductor is going to be necessarily the same at time $t = 0^-$ and $t = 0^+$. All other quantities must be analyzed at $t = 0^-$ (steady state initial conditions), and again at exactly $t = 0$ to determine the response at $t = 0^+$. Then, final steady-state conditions can be analyzed to find the response at $t \rightarrow \infty$. If the initial condition is discontinuous, the value at $t = 0^+$ should be used in the equation.

The steps for deriving an equation for the general transient response for an RL circuit follow. To reiterate, because the discharging and charging responses are just special cases of the general transient response, these steps can be followed to complete any RL circuit transient analysis.

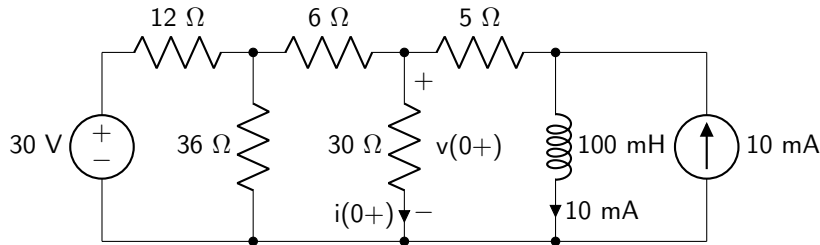
1. Calculate the initial condition(s) of the circuit (any voltage(s) or current(s) that you are asked to find).
2. If necessary, calculate any needed circuit properties at time $t = 0^+$.
3. Calculate the final condition(s) of the circuit.
4. Use the final configuration of the circuit to calculate the equivalent resistance as seen by the inductor to calculate the RL time constant.

Example: General transient response of a resistor in an RL circuit

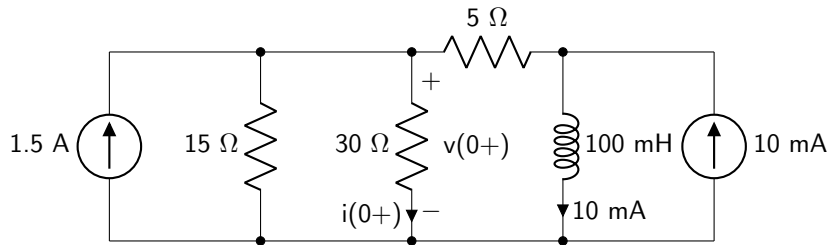
Calculate $i(t)$ and $v(t)$ for the circuit below. The switch closes at a time of 0 seconds and re-opens at a time of 0.04 seconds.



Analyze the initial conditions of the circuit, before the switch closes at 0 seconds. At this time, the switch is open and the inductor acts as a short. The current through the inductor will be 10 mA and the current through the resistor will be 0 mA. Because the current through the resistor is discontinuous at the moment the switch closes, it will be important to analyze the circuit at exactly $t = 0$ s.



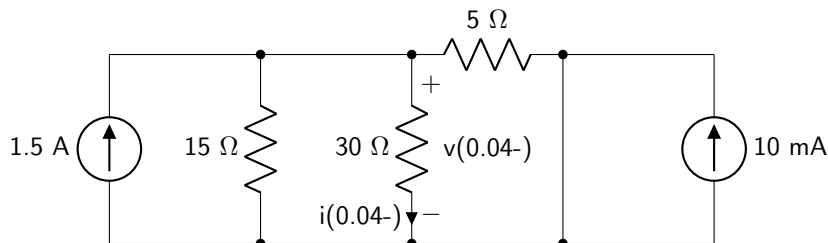
Source transformation can be used to simplify the circuit. The intermediate steps are not shown to save space.



Analyzing the node between the 5 Ω resistor, the inductor, and the 10 mA current source, no current flows through the 5 Ω resistor. This makes the calculation of $i(0^+)$ a current divider using only the 1.5 A source.

$$i(0^+) = 1500 \text{ mA} \left(\frac{30 \text{ } \Omega // 15 \text{ } \Omega}{30 \text{ } \Omega} \right) = 500 \text{ mA}$$

The same source-transformed circuit can be used to analyze the circuit as the time approaches 40 ms. In this steady-state, the inductor acts as a short.



The 10 mA source does not contribute any current to the 30 Ω resistor (as all of the resistors in that configuration are shorted by the inductor). Therefore, a current divider can be used to determine the current

through the resistor that is contributed by the 1.5 A source.

$$i(0.04^-) = 1500 \text{ mA} \left(\frac{30 \Omega // 15 \Omega // 5 \Omega}{30 \Omega} \right) = 166.67 \text{ mA}$$

It will be useful to determine the current flow through the inductor at this time, as this will be helpful in determining the current at $t = 40+$ ms through the resistor. Using superposition, we can see that a current divider can be used for the 1.5 A source, and all of the 10 mA source will be flowing through the inductor.

$$i_L(0.04) = 1500 \text{ mA} \left(\frac{30 \Omega // 15 \Omega // 5 \Omega}{5 \Omega} \right) + 10 \text{ mA} = 1010 \text{ mA}$$

The time period between 0 and 40 ms can be analyzed to find the equivalent resistance to calculate the RL time constant. All current sources can be deactivated.

$$R_{EQ} = 15 \Omega // 30 \Omega + 5 \Omega = 15 \Omega$$

The RL time constant can now be calculated.

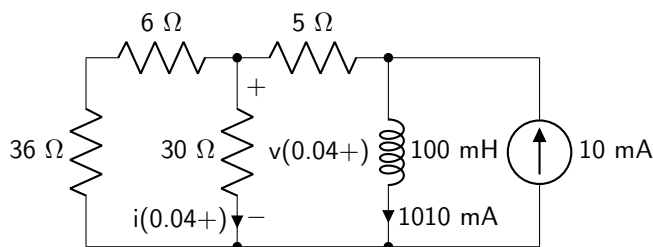
$$\tau = \frac{0.1 \text{ H}}{15 \Omega} = 0.007 \text{ s}$$

The current and voltage signals can be calculated during this time period using equation 6.14 and Ohm's law.

$$i(t) = (166.67 \text{ mA} + 333.33 \text{ mA} e^{-150t}) (u(t) - u(t - 0.04))$$

$$v(t) = (5 \text{ V} + 10 \text{ V} e^{-150t}) (u(t) - u(t - 0.04))$$

The circuit must be analyzed at the instant the switch re-opens at a time of 40 ms.



Analyzing the node between the 5 Ω resistor, the inductor, and the 10 mA current source, the 5 Ω resistor must be supplying 1 A of current to the inductor for KCL to hold. A current divider can therefore be used to calculate $i(0.04+)$.

$$i(0.04^+) = -1000 \text{ mA} \left(\frac{42 \Omega // 30 \Omega}{30 \Omega} \right) = -583.33 \text{ mA}$$

As time approaches infinity, and the circuit reaches a new steady-state, all current will flow through the inductor, and none will flow through the resistor.

The new RL time constant must be calculated. The new equivalent resistance is calculated below.

$$R_{EQ} = (36 \, \Omega + 6 \, \Omega) // 30 \, \Omega + 5 \, \Omega = 22.5 \, \Omega$$

The RL time constant can now be calculated.

$$\tau = \frac{0.1 \, \text{H}}{22.5 \, \Omega} = 0.004 \, \text{s}$$

This is a discharging scenario, so equation 6.9 can be used to find an equation for the current through the resistor. Ohm's law can be used to determine the voltage drop over the resistor during the same time period.

$$i(t) = \left(-583.33 \, \text{mA} e^{-225(t-0.04)} \right) u(t - 0.04)$$

$$v(t) = \left(-17.5 \, \text{V} e^{-225(t-0.04)} \right) u(t - 0.04)$$

Both signals can be summed together to find the total current and voltage characteristics of the resistor.

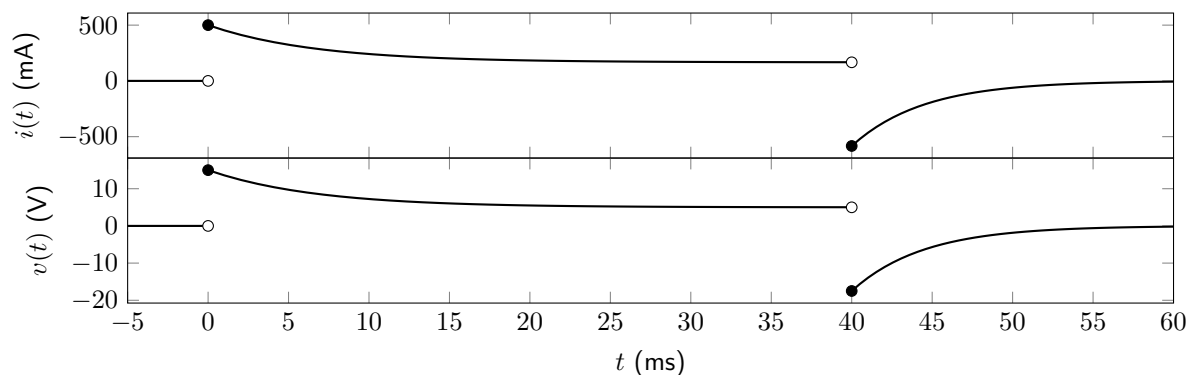
$$i(t) = (166.67 \, \text{mA} + 333.33 \, \text{mA} e^{-150t}) (u(t) - u(t - 0.04))$$

$$+ \left(-583.33 \, \text{mA} e^{-225(t-0.04)} \right) u(t - 0.04)$$

$$v(t) = (5 \, \text{V} + 10 \, \text{V} e^{-150t}) (u(t) - u(t - 0.04))$$

$$+ \left(-17.5 \, \text{V} e^{-225(t-0.04)} \right) u(t - 0.04)$$

These two signals are plotted below as functions of time.



Example Problems

Inductance and Equivalent Inductance

1. The current flowing through a 20 mH inductor is $i(t) = 50t u(t)$ mA. Derive equations for the voltage dropped over the inductor and the instantaneous power consumed by the inductor.
2. The voltage dropped over a 15 μ H inductor is $v(t) = \cos(5000t)u(t)$ V. Derive equations for the current flowing through the inductor and the instantaneous power consumed by the inductor.
3. Calculate the equivalent inductance of the circuit shown in figure 6.8.

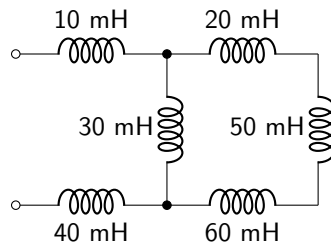


Figure 6.8: Circuit diagram for inductance and equivalent inductance question 3.

4. Calculate the equivalent inductance of the circuit shown in figure 6.9.

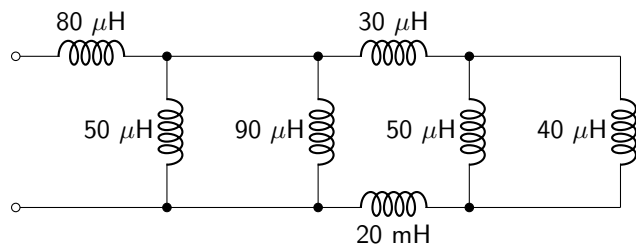


Figure 6.9: Circuit diagram for inductance and equivalent inductance question 4.

5. Determine the value of the inductor L_X given that the circuit shown in figure 6.10 has an equivalent inductance of 250 μ H.

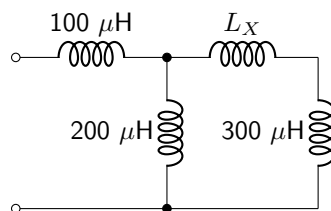


Figure 6.10: Circuit diagram for inductance and equivalent inductance question 5.

Resistor-Inductor Circuits

6. Calculate an expression for $i(t)$ and $v(t)$ given the circuit shown in figure 6.11. The switch moves from position a to b at time of zero seconds.

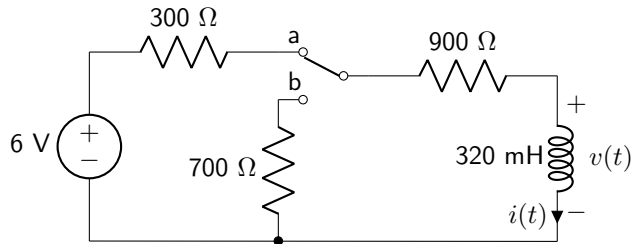


Figure 6.11: Circuit diagram for resistor-inductor circuits question 6.

7. Calculate an expression for $i(t)$ and $v(t)$ given the circuit shown in figure 6.12. The switch opens at a time of zero seconds.

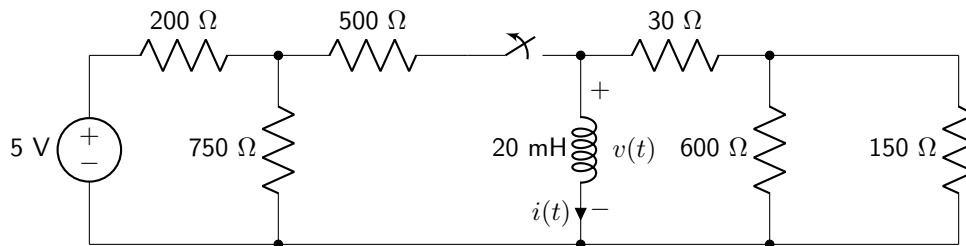


Figure 6.12: Circuit diagram for resistor-inductor circuits question 7.

8. Calculate an expression for $i(t)$ and $v(t)$ given the circuit shown in figure 6.13. The switch closes at a time of zero seconds.

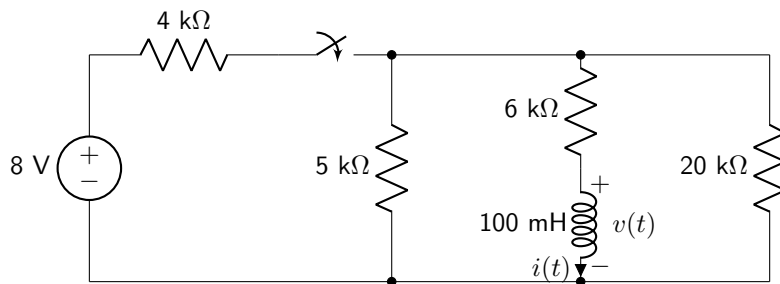


Figure 6.13: Circuit diagram for resistor-inductor circuits question 8.

9. Calculate an expression for $i(t)$ and $v(t)$ given the circuit shown in figure 6.14. The switch closes at a time of zero seconds.

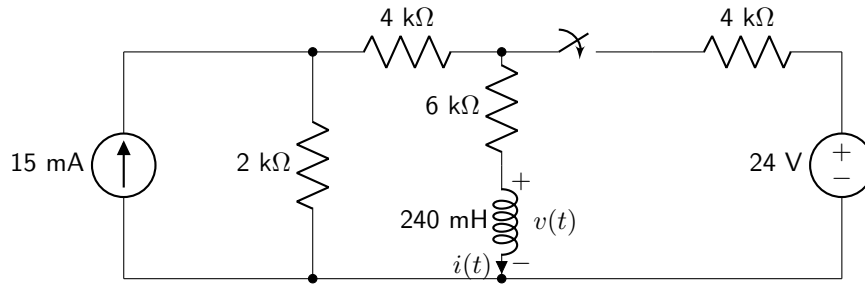


Figure 6.14: Circuit diagram for resistor-inductor circuits question 9.

10. Calculate an expression for $i(t)$ and $v(t)$ given the circuit shown in figure 6.14. Switch S1 opens at a time of zero seconds, and switch S2 closes at a time of zero seconds.

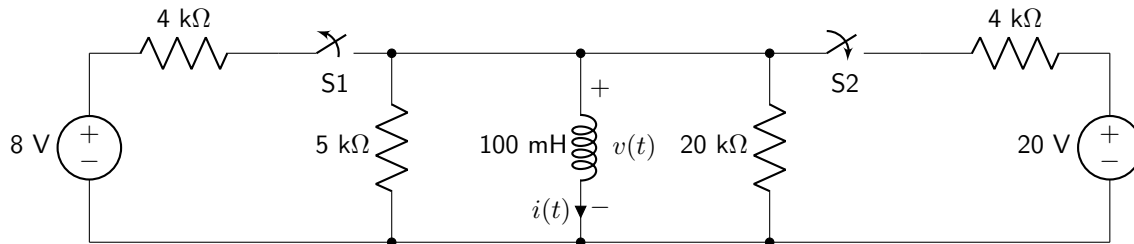


Figure 6.15: Circuit diagram for resistor-inductor circuits question 10.

7 Second Order Circuits

Second order circuits contain two equivalent energy storage elements (which could be two capacitors that cannot be combined to an equivalent capacitance, two inductors that cannot be combined to an equivalent inductance, or one equivalent capacitance and one equivalent inductance).

7.1 Homogeneous Second Order Circuit Responses

The analysis of the voltage or current in a second order circuit will be a second order, linear differential equation. The homogeneous case solves for voltage or current when all sources have been disconnected from the energy storage elements. Equation 7.1 shows the form of the second order equation for voltage. An identical equation could be solved for current. Note that the highest order term has a coefficient of one.

$$0 = \frac{d^2v(t)}{dt^2} + a_1 \frac{dv(t)}{dt} + a_0v(t) \quad (7.1)$$

The values of a_1 and a_0 can be used to solve for important circuit parameters. The damping parameter is known as α and is given by equation 7.2. The units of α are Np/s (Nepers per second). Nepers are a dimensionless quantity similar to radians. The damping parameter corresponds to how quickly the value of the voltage or current decays after the circuit elements have been removed from the power supply.

$$\alpha = \frac{a_1}{2} \quad (7.2)$$

The resonant frequency of the circuit is known as ω_0 and is given by equation 7.3. The units of ω_0 are rad/s (radians per second).

$$\omega_0 = \sqrt{a_0} \quad (7.3)$$

To solve a homogeneous second order circuit, the characteristic equation needs to be solved. The solution depends on the roots of the equation. The equations of the roots are shown in equation 7.4 and 7.5. The roots are not usually expressed with units. However, because they are multiplied by time in the exponent of an exponential function, it is important to understand that they are not dimensionless quantities.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad (7.4)$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \quad (7.5)$$

Based on the value of the roots, there are three different possible output responses that a homogeneous second order circuit can have. These are: overdamped, underdamped, and critically damped. Each will be explored in its own subsection below.

7.1.1 Overdamped

An overdamped circuit response occurs when $\alpha > \omega_0$. In an overdamped scenario, the circuit takes a long time to respond to changes in the circuit conditions. Both of the roots of the characteristic equation (s_1 and s_2) are real and unequal.

The form of the solution to an overdamped circuit is shown in equation 7.6 for voltage, but an identical form could be found for current (substituting $v(t)$ terms for $i(t)$ terms).

$$v(t) = (A_1 e^{s_1 t} + A_2 e^{s_2 t}) u(t) + v(0^-) u(-t) \quad (7.6)$$

All second order circuits with a positive resistance will have negative roots, indicating a stable response to a change in conditions (all exponential terms will converge rather than diverge over time). (Circuit stability is discussed in section 9.4.2.)

The coefficients A_1 and A_2 (which have units of voltage or current, depending on what is being solved) can be derived by solving equation 7.6 for $v(0^+)$ and $v'(0^+)$. This derivation will not be shown, but the solution for both coefficients is given in equation 7.7 and 7.8. Again, if current is being solved for, all of the voltage terms will be replaced with current terms.

$$A_1 = \frac{v(0^+)s_2 - v'(0^+)}{s_2 - s_1} \quad (7.7)$$

$$A_2 = \frac{v'(0^+) - v(0^+)s_1}{s_2 - s_1} \quad (7.8)$$

A graph depicting the voltage response of an overdamped circuit is shown in figure 7.1. A similar graph could be made for a current response.

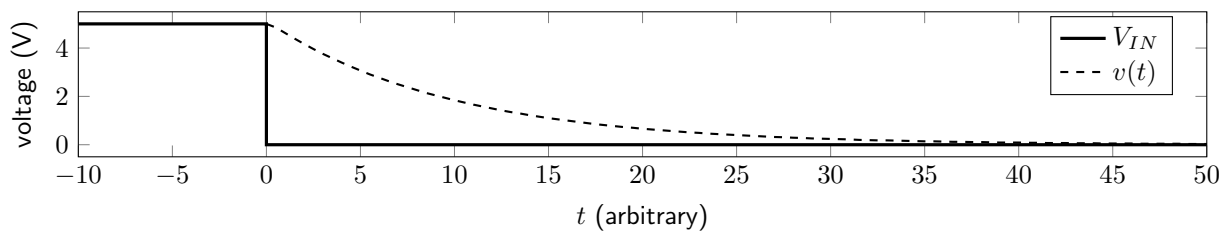


Figure 7.1: Voltage response of an overdamped second order circuit to an abrupt removal of input voltage.

7.1.2 Critically Damped

A critically damped circuit response occurs when $\alpha = \omega_0$. The critically damped circuit has the quickest approach to the steady-state value without oscillating. (In other words, it is the boundary condition between an overdamped and an underdamped circuit.) Both of the roots of the characteristic equation (s_1 and s_2) are real and equal.

The form of the solution to a critically damped circuit is shown in equation 7.9 for voltage. An identical form could be found for current.

$$v(t) = (A_1 e^{-\alpha t} + A_2 t e^{-\alpha t}) u(t) + v(0^-) u(-t) \quad (7.9)$$

The coefficient A_1 has units of V (or A) and the coefficient A_2 has units of V/s (or A/s). Both coefficients can be derived by solving equation 7.9 for $v(0)$ and $v'(0)$. This derivation will not be shown. The solution for both coefficients is given in equation 7.10 and 7.11. To find the coefficients for current, all voltage terms should be replaced with current terms.

$$A_1 = v(0^+) \quad (7.10)$$

$$A_2 = v'(0^+) + \alpha v(0^+) \quad (7.11)$$

A graph depicting the voltage response of a critically damped circuit is shown in figure 7.2. A similar graph could be made for a current response.

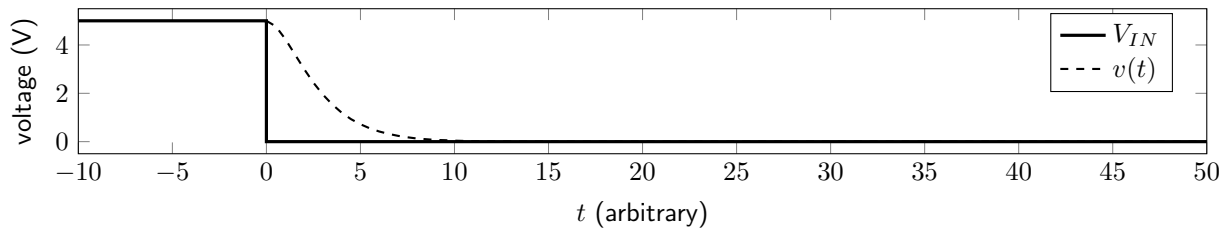


Figure 7.2: Voltage response of a critically damped second order circuit to an abrupt removal of input voltage.

7.1.3 Underdamped

An underdamped circuit response occurs when $\alpha < \omega_0$. In an underdamped scenario, the circuit responds so quickly to changes in the circuit conditions that the output oscillates around the final value, eventually being attenuated to zero. The roots of the characteristic equation (s_1 and s_2) are complex conjugates.

The form of the solution to an underdamped circuit is shown in equation 7.12 for voltage. An identical form could be found for current.

$$v(t) = e^{-\alpha t} [B_1 \cos \beta t + B_2 \sin \beta t] u(t) + v(0^-) u(-t) \quad (7.12)$$

Note a new parameter, β , which is equal to the oscillation frequency and has units of rad/s. The equation for the oscillation frequency is given in equation 7.13.

$$\beta = \sqrt{\omega_0^2 - \alpha^2} \quad (7.13)$$

The coefficients B_1 and B_2 have units of V (or A) and can be derived by solving equation 7.12 for $v(0)$ and $v'(0)$. This derivation will not be shown in this textbook. The solution for the coefficients is given in equation 7.14 and 7.15. To find the coefficients for current, all voltage terms should be replaced with current terms.

$$B_1 = v(0^+) \quad (7.14)$$

$$B_2 = \frac{v'(0^+) + \alpha v(0^+)}{\beta} \quad (7.15)$$

A graph depicting the voltage response of an underdamped circuit is shown in figure 7.3. A similar graph could be made for a current response.

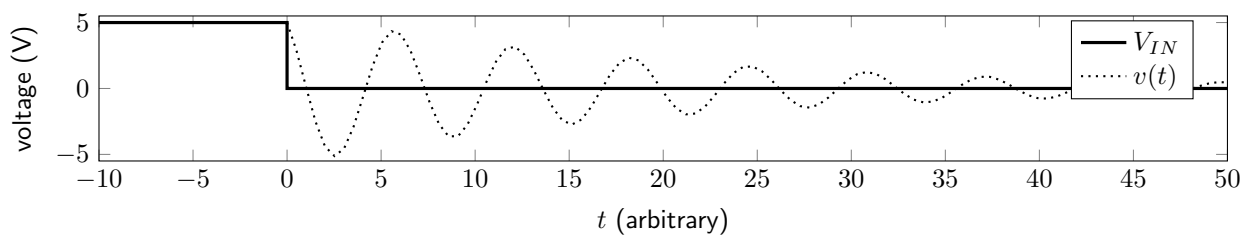


Figure 7.3: Voltage response of an underdamped second order circuit to an abrupt removal of input voltage.

7.2 Homogeneous Series RLC Circuit

A homogeneous series RLC circuit is configured such that, when the input voltage is removed, the circuit components contain one or more resistor, capacitor, and inductor, all in series with each other. An example of a series RLC circuit is shown in figure 7.4. The switch is in position a for a long time before moving to position b at time $t = 0$.

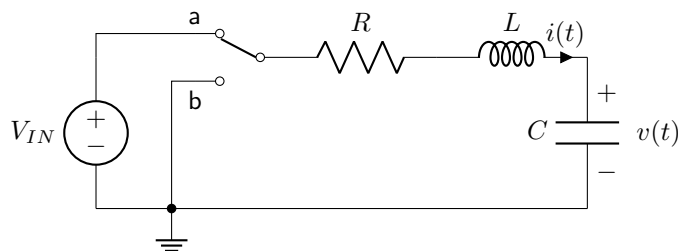


Figure 7.4: A series RLC circuit.

First, the initial conditions of the circuit should be found. In this particular circuit, $v(0) = V_{IN}$. Equation 5.5 can be solved for $t = 0^+$ to find the initial first derivative of the voltage. Because this is a second order circuit, both the initial voltage and the initial first derivative of voltage must be calculated to obtain a complete solution to the equation. For series RLC circuits, equation 5.5 can be rewritten as shown in equation 7.16. (The circuit in figure 7.4 has $v'(0^+) = 0$, but that is not necessarily true for all series RLC

circuits.)

$$v'(0^+) = \frac{i(0^+)}{C} \quad (7.16)$$

Kirchhoff's laws can then be used to solve for the voltage drop over the capacitor ($v(t)$) once the switch has been moved to position b. The KVL equation is shown in equation 7.17.

$$0 = Ri(t) + L \frac{di(t)}{dt} + v(t) \quad (7.17)$$

Equation 5.5 relating the current through a capacitor to the voltage drop over a capacitor can then be plugged into equation 7.17 to find the second order equation. This derivation is shown in equation 7.18.

$$0 = \frac{d^2v(t)}{dt^2} + \frac{R}{L} \frac{dv(t)}{dt} + \frac{1}{LC}v(t) \quad (7.18)$$

Now that the equation is configured to have the second order derivative term with a coefficient of one, it is easy to determine the values of a_1 (R/L) and a_0 ($1/LC$). The damping parameter α is obtained by using equation 7.2 and is shown in equation 7.19.

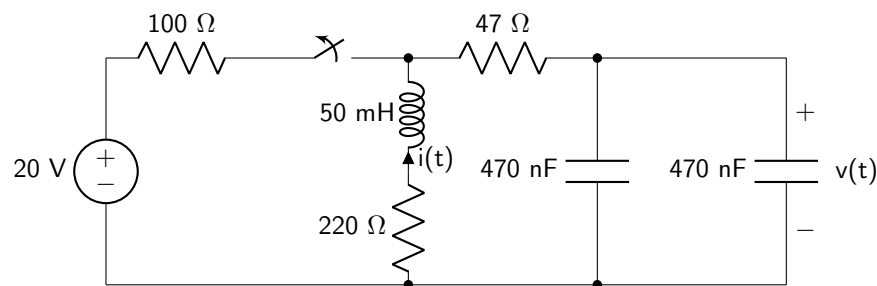
$$\alpha = \frac{R}{2L} \quad (7.19)$$

The resonant frequency ω_0 is obtained using equation 7.3 and is shown in equation 7.20.

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad (7.20)$$

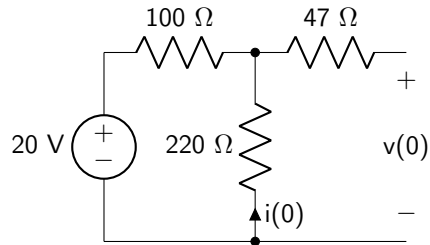
Example: Homogeneous RLC series circuit

Calculate $v(t)$ for the circuit below. The switch opens at a time of 0 seconds.



Analyze the initial conditions of the circuit, before the switch opens at 0 seconds. The inductor can be treated as a short and the capacitors can be treated as open circuits. The voltage over the capacitor and current through the inductor are continuous at time $t = 0$ and do not need to be separately analyzed at

$t = 0^+$.



Use Ohm's law to calculate the initial current flow.

$$i(0) = \frac{-20\text{V}}{320\ \Omega} = -62.5\ \text{mA}$$

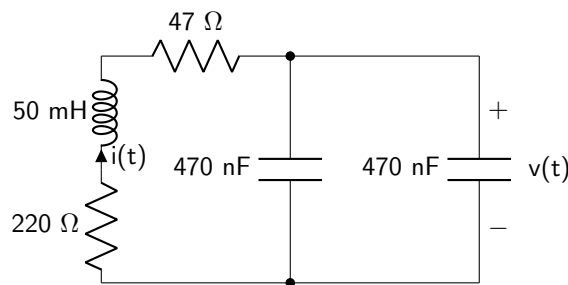
Use a voltage divider to calculate the initial voltage drop.

$$v(0) = 20\ \text{V} \left(\frac{220\ \Omega}{320\ \Omega} \right) = 13.75\ \text{V}$$

Use equation 7.16 to determine the initial derivative of the voltage.

$$v'(0^+) = \frac{-0.0625\ \text{A}}{940 \times 10^{-9}\ \text{F}} = -66489.36\ \text{V/s}$$

At this point, the circuit can be analyzed with the switch open to determine the values of R, L, and C to use to determine the damping parameter and resonant frequency.



The total resistance is equal to $267\ \Omega$, the inductance is $50\ \text{mH}$, and the capacitance is $940\ \text{nF}$. Use equation 7.19 to calculate the damping parameter.

$$\alpha = \frac{267\ \Omega}{2 \times 0.05\ \text{H}} = 2670\ \text{Np/s}$$

Use equation 7.20 to calculate the resonant frequency.

$$\omega_0 = \frac{1}{\sqrt{(0.05\ \text{H})(940 \times 10^{-9}\ \text{F})}} = 4612.66\ \text{rad/s}$$

This circuit is underdamped. Use equation 7.13 to calculate the oscillation frequency of the circuit.

$$\beta = \sqrt{4612.66^2 - 2670^2} = 3761.35 \text{ rad/s}$$

Use equations 7.14 and 7.15 to calculate the coefficients.

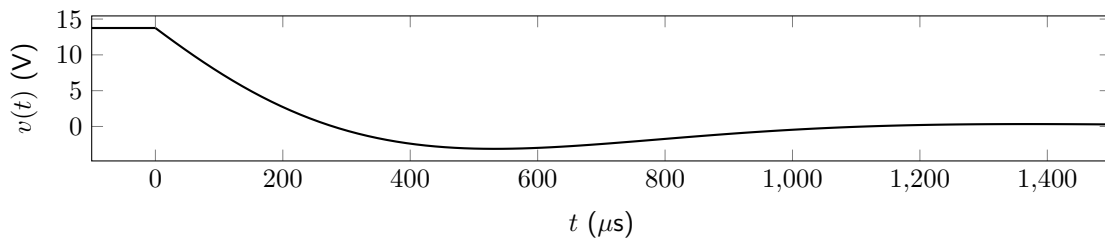
$$B_1 = 13.75 \text{ V}$$

$$B_2 = \frac{-66489.36 + (2670)(13.75)}{3761.35} = -7.92 \text{ V}$$

Plug all of these quantities into equation 7.12 to calculate the equation for $v(t)$.

$$v(t) = e^{-2670t} [13.75 \text{ V} \cos(3761.35t) - 7.92 \text{ V} \sin(3761.35t)] u(t) + 13.75 \text{ V} u(-t)$$

The voltage is plotted below as a function of time.



7.3 Homogeneous Parallel RLC Circuit

A homogeneous parallel RLC circuit is configured such that, when the input current is removed, the circuit components contain resistance, inductance, and capacitance in parallel with each other. An example of a parallel RLC circuit is shown in figure 7.5. The switch is in position a for a long time before moving to position b at time $t = 0$.

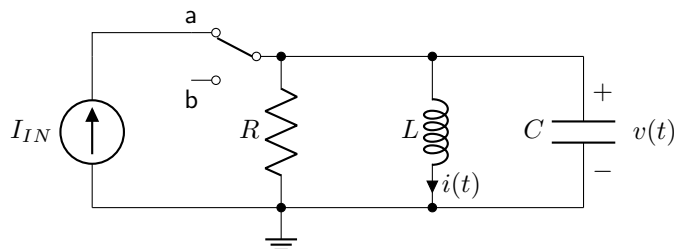


Figure 7.5: A parallel RLC circuit.

First, the initial conditions of the circuit should be found. In this particular circuit, $i(0) = I_{IN}$. The equation for voltage in equation 6.4 can be solved for $t = 0^+$ to find the initial first derivative of the current. This is shown in equation 7.21 for RLC parallel circuits. (The circuit in figure 7.5 has $i'(0^+) = 0$, but that

is not necessarily true for all parallel RLC circuits, particularly those that have a capacitor that is not fully discharged.)

$$i'(0^+) = \frac{v(0^+)}{L} \quad (7.21)$$

Kirchhoff's laws can then be used to solve for the current flow through the inductor ($i(t)$) once the switch has been moved to position b. The KCL equation is shown in equation 7.22.

$$0 = \frac{v(t)}{R} + i(t) + C \frac{dv(t)}{dt} \quad (7.22)$$

Equation 6.4 relating the voltage drop over an inductor to the current can then be plugged into equation 7.22 to find the second order equation. This derivation is shown in equation 7.23.

$$0 = \frac{d^2 i(t)}{dt^2} + \frac{1}{RC} \frac{di(t)}{dt} + \frac{1}{LC} i(t) \quad (7.23)$$

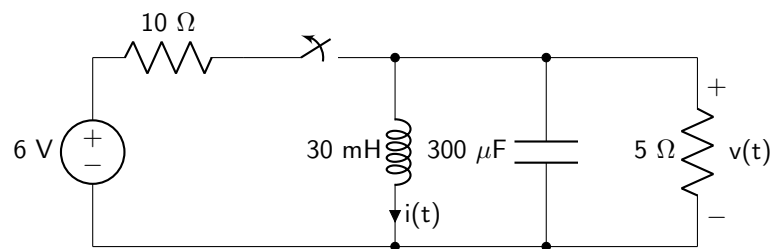
Now that the equation is configured to have the second order term with a coefficient of one, it is easy to determine the values of a_1 ($1/RC$) and a_0 ($1/LC$). The damping parameter α is obtained by using equation 7.2 and is shown in equation 7.24.

$$\alpha = \frac{1}{2RC} \quad (7.24)$$

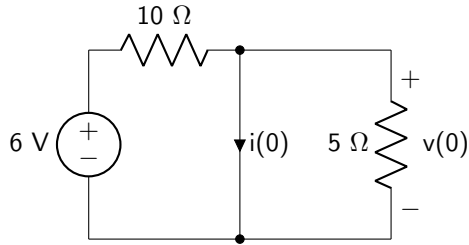
The resonant frequency ω_0 is obtained using equation 7.3. Because a_0 in the parallel RLC circuit is the same as a_0 in the series RLC circuit, they will have the same resonant frequency, given by equation 7.20.

Example: Homogeneous RLC parallel circuit

Calculate $i(t)$ for the circuit below. The switch opens at a time of 0 seconds.



Analyze the initial conditions of the circuit, before the switch opens at 0 seconds. The inductor can be treated as a short and the capacitor can be treated as an open circuit. The voltage dropped over the capacitor and current flowing through the inductor are both continuous at $t = 0$.

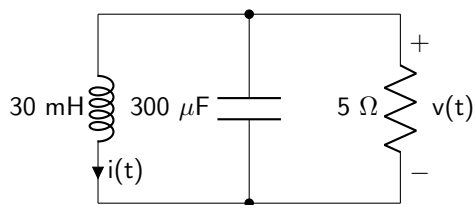


Use Ohm's law to calculate the initial current flow.

$$i(0) = \frac{6\text{V}}{10\ \Omega} = 0.6\ \text{A}$$

Because there is no voltage dropped over the $5\ \Omega$ resistor (it is shorted out by the inductor), the initial voltage drop will be zero, and the value of $i'(0^+)$ will also be zero.

At this point, the circuit can be analyzed with the switch open to determine the values of R, L, and C to use to determine the damping parameter and resonant frequency.



The total resistance is equal to $5\ \Omega$, the inductance is $30\ \text{mH}$, and the capacitance is $300\ \mu\text{F}$. Use equation 7.24 to calculate the damping parameter.

$$\alpha = \frac{1}{2(5\ \Omega)(300 \times 10^{-6}\ \text{F})} = 333.33\ \text{Np/s}$$

Use equation 7.20 to calculate the resonant frequency.

$$\omega_0 = \frac{1}{\sqrt{(0.03\ \text{H})(300 \times 10^{-6}\ \text{F})}} = 333.33\ \text{rad/s}$$

This circuit is critically damped with both roots equal to -333.33 . Use equations 7.10 and 7.11 to calculate the coefficients.

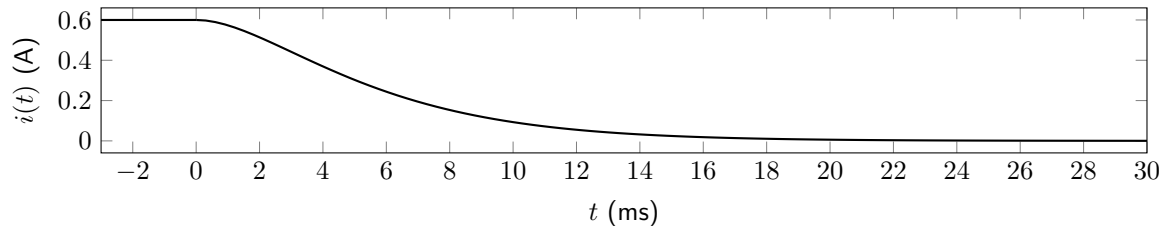
$$A_1 = 0.6\ \text{A}$$

$$A_2 = (333.33)(0.6) = 200\ \text{A/s}$$

Plug all of these quantities into equation 7.9 to calculate the equation for $i(t)$.

$$i(t) = (0.6 \text{ A } e^{-333.33t} + 200 \text{ A/s } t e^{-333.33t}) u(t) + 0.6 \text{ A } u(-t)$$

The current is plotted below as a function of time.



7.3.1 Analysis with Parasitic Resistance

The analysis above ignores the parasitic resistance that exists in real inductors. A realistic circuit, which explicitly includes the parasitic resistance in series with the inductor, can be analyzed. The corresponding circuit diagram is shown in figure 7.6.

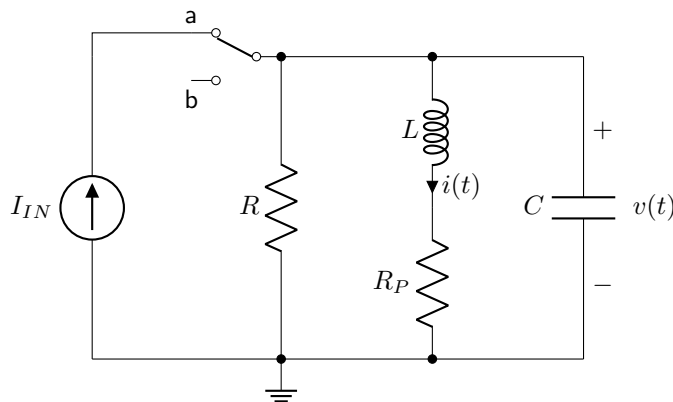


Figure 7.6: A parallel RLC circuit including parasitic resistance R_P that comes from the inductor.

As will be discussed in section 7.4, second order circuits in general are analyzed by determining initial conditions, and then using Kirchhoff's laws to find a second order equation for the quantity of interest. That is what will be done in this circuit.

KCL can be used to find the first equation, which is identical to that of the non-parasitic RLC circuit, given in equation 7.22. KVL around the loop containing the inductor, parasitic resistance, and capacitor is given in equation 7.25.

$$v(t) = L \frac{di(t)}{dt} + R_P i(t) \quad (7.25)$$

Equation 7.25 can be plugged into equation 7.22 to solve for the second order equation in terms of current.

The full derivation will not be shown. The final result is given in equation 7.26.

$$0 = \frac{d^2 i(t)}{dt^2} + \left[\frac{1}{RC} + \frac{R_P}{L} \right] \frac{di(t)}{dt} + \left[\frac{1}{LC} + \frac{R_P}{RLC} \right] i(t) \quad (7.26)$$

Now that the equation is configured to have the second order term with a coefficient of one, it is easy to determine the values of a_1 and a_0 . The damping parameter α can be solved using equation 7.2 and is shown in equation 7.27.

$$\alpha = \frac{1}{2RC} + \frac{R_P}{2L} \quad (7.27)$$

The resonant frequency ω_0 can be solved using equation 7.3 and is shown in equation 7.28.

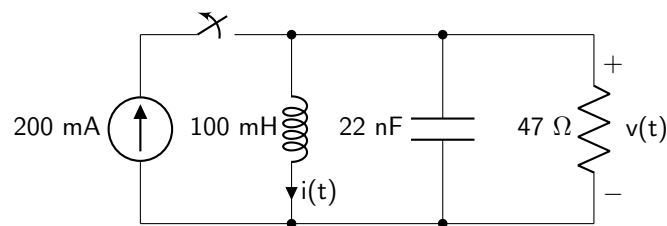
$$\omega_0 = \sqrt{\frac{1}{LC} + \frac{R_P}{RLC}} \quad (7.28)$$

Equation 7.25 can also be solved at time $t = 0$ to find the initial condition of the first derivative of the current, shown in equation 7.29.

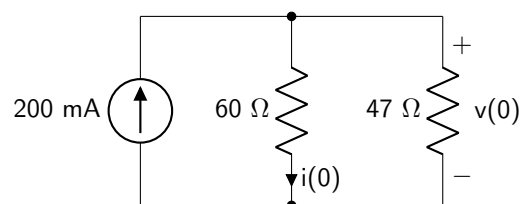
$$i'(0^+) = \frac{v(0^+) - R_P i(0^+)}{L} \quad (7.29)$$

Example: Homogeneous RLC parallel circuit with parasitic resistance

Calculate $i(t)$ for the circuit below. The switch opens at a time of 0 seconds. The inductor has a parasitic resistance of 60Ω (not shown).



Analyze the initial conditions of the circuit, before the switch opens at 0 seconds. The voltage dropped over the capacitor and current flowing through the inductor will both be continuous at $t = 0$. The inductor can be treated as a short and the capacitor can be treated as an open circuit. The parasitic resistance is shown in this diagram to make it clear that the inductor is not shorting out the 47Ω resistor.



Use a current divider to calculate the initial current flow.

$$i(0) = 200 \text{ mA} \left(\frac{60 \Omega // 47 \Omega}{60 \Omega} \right) = 87.85 \text{ mA}$$

The initial voltage drop will not be zero. Use Ohm's law to calculate $v(0)$.

$$v(0) = (0.2 \text{ A})(60 \Omega // 47 \Omega) = 5.27 \text{ V}$$

Use equation 7.29 to calculate the initial first derivative of the current. It is equal to zero. Use equation 7.27 to calculate the damping parameter.

$$\alpha = \frac{1}{2(47 \Omega)(22 \times 10^{-9} \text{ F})} + \frac{60 \Omega}{2(0.1 \text{ H})} = 483858.99 \text{ Np/s}$$

Use equation 7.28 to calculate the resonant frequency.

$$\omega_0 = \sqrt{\frac{1}{(0.1 \text{ H})(22 \times 10^{-9} \text{ F})} + \frac{60 \Omega}{(47 \Omega)(0.1 \text{ H})(22 \times 10^{-9} \text{ F})}} = 32168.56 \text{ rad/s}$$

This circuit is overdamped. Use equations 7.4 and 7.5 to calculate the roots of the characteristic equation.

$$s_1 = -483858.99 + \sqrt{483858.99^2 - 32168.56^2} = -1070.52$$

$$s_2 = -483858.99 - \sqrt{483858.99^2 - 32168.56^2} = -966647.46$$

Use equations 7.7 and 7.8 to calculate the coefficients of the equation.

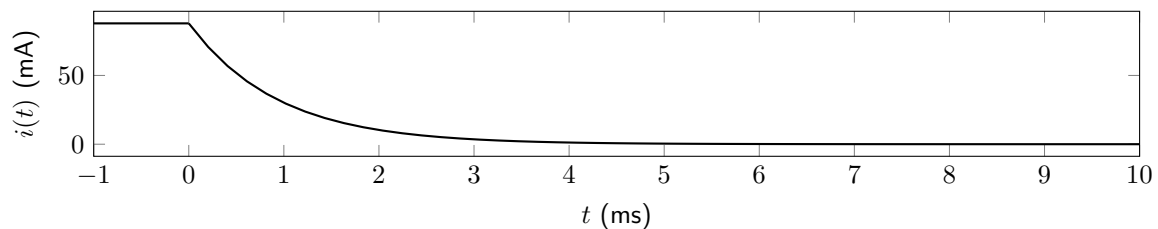
$$A_1 = \frac{(87.85)(-966647.46)}{-966647.46 + 1070.52} = 87.95 \text{ mA}$$

$$A_2 = \frac{-(87.85)(-1070.52)}{-966647.46 + 1070.52} = -0.10 \text{ mA}$$

Plug all of these quantities into equation 7.6 to calculate the equation for $i(t)$.

$$i(t) = (87.95 \text{ mA} e^{-1070.52t} - 0.10 \text{ mA} e^{-966647.46t}) u(t) + 87.85 \text{ mA} u(-t)$$

The current is plotted below as a function of time.



7.4 General Homogeneous Second Order Circuits

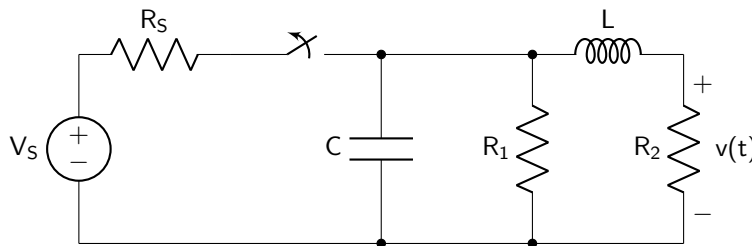
The procedure for solving second order circuits that do not follow the exact format of parallel RLC or series RLC follows.

1. Use KCL and KVL to find a second order differential equation in terms of the desired quantity.
2. The equations from step one can be solved for $v'(0^+)$ or $i'(0^+)$.
3. Solve the circuit for the initial conditions.
4. Use the differential equation from step one to calculate α and ω_0 .
5. Calculate the coefficients.
6. Find the equation for $v(t)$ or $i(t)$.

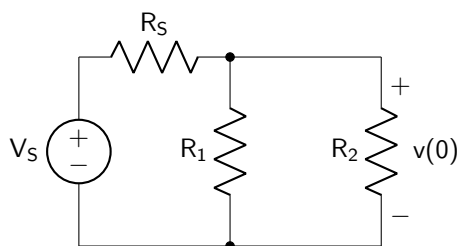
All of the equations in this section of the text have discussed finding the voltage drop over the capacitor or current flow through an inductor. It is also possible to derive a second order equation to find voltage over or current through any arbitrary component in a circuit. It is important to note that the initial conditions may be discontinuous. All other quantities must be analyzed at $t = 0^-$ (steady state initial conditions), and again at exactly $t = 0$ to determine the response at $t = 0^+$. Then, final steady-state conditions can be analyzed to find the response at $t \rightarrow \infty$. If the initial condition is discontinuous, the value at $t = 0^+$ should be used as the value of $v(0)$ or $i(0)$.

Example: Homogeneous general second order circuit in symbolic form

Calculate $v(t)$ for the circuit below. The switch opens at a time of 0 seconds.



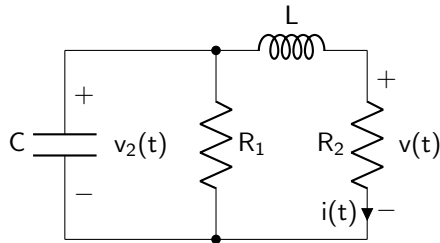
First, analyze the initial conditions of the circuit, before the switch opens at 0 seconds. The inductor can be treated as a short and the capacitor can be treated as an open circuit.



Use a voltage divider to calculate the initial voltage. This value will be continuous at time $t = 0$ s due to the inductor current and capacitor voltage drop being continuous at that moment.

$$v(0) = V_S \left(\frac{R_2 // R_1}{R_S + R_2 // R_1} \right)$$

Analyze the circuit with the switch open to determine the second order equation. Some other intermediate terms ($i(t)$ and $v_2(t)$) will be introduced to assist in the analysis.



Perform KVL around the loop consisting of the capacitor, inductor, and resistor R_2 .

$$v_2(t) = L \frac{di(t)}{dt} + v(t)$$

The current flowing through the inductor ($i(t)$) is equal to $v(t)/R_2$. (Note that this equation will be used below to find $v'(0)$.)

$$v_2(t) = \frac{L}{R_2} \frac{dv(t)}{dt} + v(t)$$

Perform KCL at the node connecting the capacitor, resistor R_1 , and the inductor.

$$\frac{v(t)}{R_2} + C \frac{dv_2(t)}{dt} + \frac{v_2(t)}{R_1} = 0$$

Solve the last two equations for $v(t)$ and ensure that the highest order term has a coefficient of one.

$$0 = \frac{d^2v(t)}{dt^2} + \left[\frac{R_2}{L} + \frac{1}{CR_1} \right] \frac{dv(t)}{dt} + \left[\frac{R_2}{R_1LC} + \frac{1}{LC} \right] v(t)$$

Solve for alpha and omega.

$$\alpha = \frac{R_2}{2L} + \frac{1}{2CR_1}$$

$$\omega_0 = \sqrt{\frac{R_2}{R_1LC} + \frac{1}{LC}}$$

Find an equation for the initial derivative of the voltage drop.

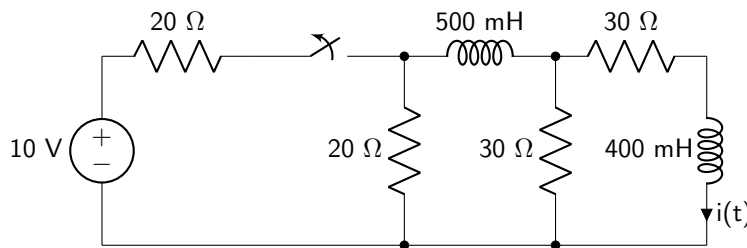
$$v'(0^+) = \frac{R_2}{L} (v_2(0) - v(0))$$

Because the initial voltage drops $v(0)$ and $v_2(0)$ are equal, this value will be equal to zero, regardless of the value of any individual components.

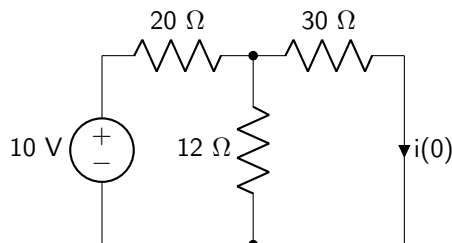
This circuit can now be solved given any values of V_S , R_S , L , C , R_1 and R_2 .

Example: Homogeneous general second order circuit in numeric form

Calculate $i(t)$ for the circuit below. The switch opens at a time of 0 seconds.



Analyze the initial conditions of the circuit, before the switch opens at 0 seconds. Both inductors can be treated as short circuits. Because the 500 mH inductor acts as a short, the 20 Ω and 30 Ω resistors are in parallel and have been reduced to an equivalent resistance.



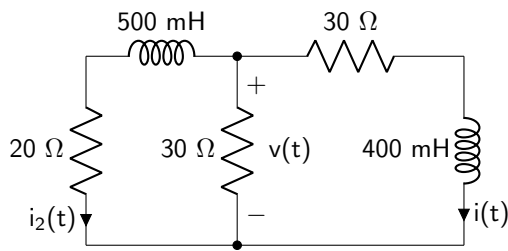
Use a current divider to calculate the initial current. Because this is the current through an inductor, it will be continuous at time $t = 0$ s.

$$i(0) = \frac{10 \text{ V}}{20 \Omega} \left(\frac{20 \Omega // 12 \Omega // 30 \Omega}{30 \Omega} \right) = 0.1 \text{ A}$$

It will also be useful to determine the initial voltage drop over the 12 Ω resistor. Use a voltage divider to calculate this quantity. This quantity will also be continuous at time $t = 0$ s. If it weren't, the circuit would need to be analyzed at exactly the moment the switch was opened to determine the value at $t = 0^+$.

$$v_{12 \Omega}(0) = 10 \text{ V} \left(\frac{12 \Omega // 30 \Omega}{20 \Omega + 12 \Omega // 30 \Omega} \right) = 3 \text{ V}$$

Analyze the circuit with the switch open to determine the second order equation. Some other intermediate terms ($v(t)$ and $i_2(t)$) will be introduced to assist in the analysis.



Perform KVL around the loop consisting of both $30\ \Omega$ resistors and the $400\ \text{mH}$ inductor. At this point, all units will be removed from equations. All quantities will be in terms of Ω , V, A, and H.

$$v(t) = 30i(t) + 0.4 \frac{di(t)}{dt}$$

Perform KCL at the node connecting the $500\ \text{mH}$ inductor and both $30\ \Omega$ resistors.

$$i_2(t) = -\frac{v(t)}{30} - i(t)$$

One last equation is required. Perform KVL around the loop consisting of the $20\ \Omega$ resistor, the $500\ \text{mH}$ inductor, and the $30\ \Omega$ resistor.

$$v(t) = 20i_2(t) + 0.5 \frac{di_2(t)}{dt}$$

There are now three equations and three unknowns. Solve each equation to find a second order equation in terms of $i(t)$. Ensure that the highest order term has a coefficient of one.

$$0 = \frac{d^2i(t)}{dt^2} + 250 \frac{di(t)}{dt} + 10500i(t)$$

Calculate alpha and omega.

$$\alpha = \frac{250}{2} = 125\ \text{Np/s}$$

$$\omega_0 = \sqrt{10500} = 102.47\ \text{rad/s}$$

This circuit is overdamped.

Find an equation for the initial derivative of the current. $v(0)$ is equal to the initial voltage drop over the

12 Ω resistor that was calculated in a previous step to be 3 V.

$$\begin{aligned} i'(0^+) &= \frac{v(0) - 30i(0)}{0.4} \\ &= \frac{3 - 30(0.1)}{0.4} \\ &= 0 \text{ A/s} \end{aligned}$$

Use equations 7.4 and 7.5 to calculate the roots of the characteristic equation.

$$\begin{aligned} s_1 &= -125 + \sqrt{125^2 - 102.47^2} = -53.41 \\ s_2 &= -125 - \sqrt{125^2 - 102.47^2} = -196.59 \end{aligned}$$

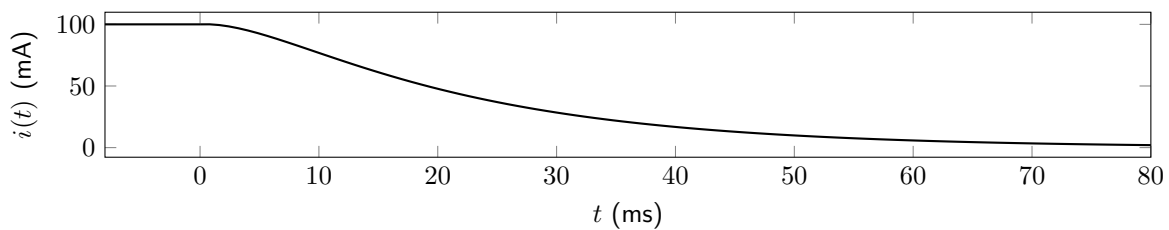
Use equations 7.7 and 7.8 to calculate the coefficients of the equation.

$$\begin{aligned} A_1 &= \frac{(0.1)(-196.59)}{-196.59 + 53.41} = 0.14 \text{ A} \\ A_2 &= \frac{-(0.1)(-53.41)}{-196.59 + 53.41} = -0.04 \text{ A} \end{aligned}$$

Plug all of these quantities into equation 7.6 to obtain the equation for $i(t)$.

$$i(t) = (0.14 \text{ A } e^{-53.41t} - 0.04 \text{ A } e^{-196.59t}) u(t) + 0.1 \text{ A } u(-t)$$

The current is plotted below as a function of time.



7.5 Non-Homogeneous Second Order Circuits

A non-homogeneous second order circuit contains a source in the final configuration of the circuit. The differential equation, which was shown in equation 7.1 for a homogeneous circuit, will no longer be equal to zero. The new form of the second order equation is shown in equation 7.30.

$$y(t) = \frac{d^2v(t)}{dt^2} + a_1 \frac{dv(t)}{dt} + a_0v(t) \quad (7.30)$$

To solve a non-homogeneous second order equation, first find the general solution. Then, find the par-

ticular solution. These two solutions are added together to find the overall circuit response. The general solution is the response of the homogeneous equation, which was discussed in the previous sections of this chapter. The particular solution $v_p(t)$ is the only new aspect to solving a non-homogeneous second order circuit.

The other change that happens is that the form of each circuit response equation (overdamped, underdamped, and critically damped) will be modified. Note that the resulting equations are the general form of an equation. The homogenous response is simply the specific case where $v_p(t)$ or $i_p(t)$ is equal to zero. The non-homogeneous equations can therefore be used to analyze any second order circuit. All of the equations given below will be in terms of voltage. To modify the equations to solve for current, simply replace all voltage terms with current terms.

The overdamped circuit will be characterized by the equation shown in equation 7.31.

$$v(t) = (v_p(t) + A_1 e^{s_1 t} + A_2 e^{s_2 t}) u(t) + v(0^-) u(-t) \quad (7.31)$$

The critically damped circuit will be characterized by the equation shown in equation 7.32.

$$v(t) = (v_p(t) + A_1 e^{-\alpha t} + A_2 t e^{-\alpha t}) u(t) + v(0^-) u(-t) \quad (7.32)$$

The underdamped circuit will be characterized by the equation shown in equation 7.33.

$$v(t) = (v_p(t) + e^{-\alpha t} [B_1 \cos \beta t + B_2 \sin \beta t]) u(t) + v(0^-) u(-t) \quad (7.33)$$

7.5.1 Constant Input Non-Homogeneous Second Order Circuits

A constant input second order circuit will have a DC source connected in the final circuit configuration. In this case, the particular solution will simply be equal to the final steady-state voltage (or current), shown in equation 7.34.

$$v_p(t) = v(\infty) \quad (7.34)$$

The coefficients for each type of circuit response can now be solved in terms of the initial conditions and final conditions of the circuit. The overdamped coefficients are given in equations 7.35 and 7.36.

$$A_1 = \frac{s_2(v(0^+) - v(\infty)) - v'(0^+)}{s_2 - s_1} \quad (7.35)$$

$$A_2 = \frac{v'(0^+) - s_1(v(0^+) - v(\infty))}{s_2 - s_1} \quad (7.36)$$

The critically damped coefficients are given in equations 7.37 and 7.38.

$$A_1 = v(0^+) - v(\infty) \quad (7.37)$$

$$A_2 = v'(0^+) - s_1(v(0^+) - v(\infty)) \quad (7.38)$$

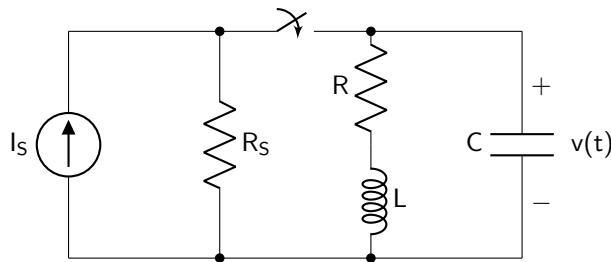
The underdamped coefficients are given in equations 7.39 and 7.40.

$$B_1 = v(0^+) - v(\infty) \quad (7.39)$$

$$B_2 = \frac{v'(0^+) + \alpha(v(0^+) - v(\infty))}{\beta} \quad (7.40)$$

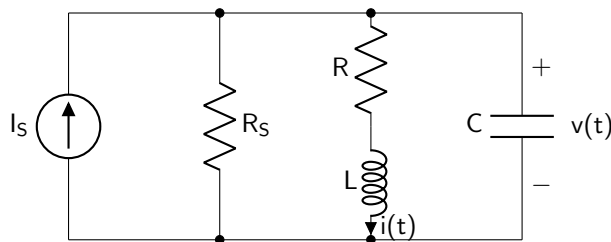
Example: Constant input non-homogeneous general second order circuit in symbolic form

Calculate $v(t)$ for the circuit below. The switch closes at a time of 0 seconds.



The initial conditions are straightforward to determine. The voltage drop will be zero as it will be disconnected from the current source and all energy will have discharged through the resistor and inductor. The voltage dropped over the capacitor and the current flowing through the inductor will both be continuous at $t = 0$.

Analyze the circuit with the switch closed to find a second order differential equation. An intermediate term $i(t)$ will be introduced to assist in the analysis.



Perform KVL around the loop consisting of the resistor R , inductor, and capacitor.

$$v(t) = Ri(t) + L \frac{di(t)}{dt}$$

Solve KCL at the node connecting the power supply, both resistors, and the capacitor.

$$I_S = \frac{v(t)}{R_s} + i(t) + C \frac{dv(t)}{dt}$$

Solve the latter equation for $i(t)$ and plug into the former to find the differential equation. Ensure that the highest order term has a coefficient of one.

$$\frac{R}{LC} I_S = \frac{d^2 v(t)}{dt^2} + \left[\frac{1}{R_s C} + \frac{R}{L} \right] \frac{dv(t)}{dt} + \left[\frac{1}{LC} + \frac{R}{R_s LC} \right] v(t)$$

The latter equation can also be used to find an equation for the initial derivative of the voltage drop.

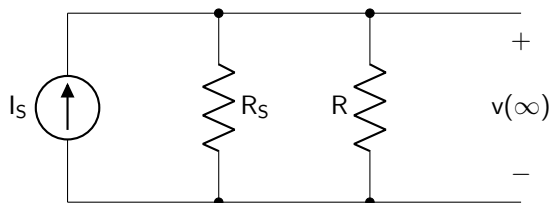
$$v'(0^+) = \frac{I_S - \frac{v(0)}{R_s} - i(0)}{C}$$

Calculate alpha and omega.

$$\alpha = \frac{R}{2L} + \frac{1}{2CR_s}$$

$$\omega_0 = \sqrt{\frac{R}{R_s LC} + \frac{1}{LC}}$$

The final value of the voltage drop over the capacitor will be used to determine the particular solution $v_p(t)$. The inductor can be replaced with a short and the capacitor can be replaced with an open.



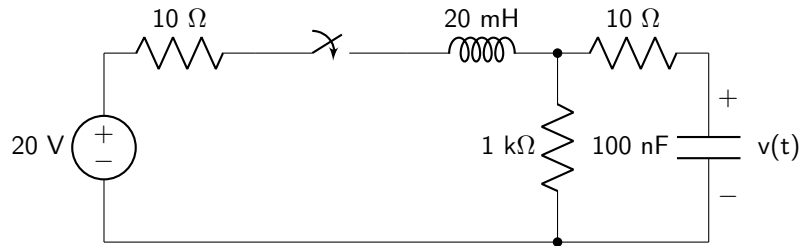
Use Ohm's law to calculate this value.

$$v(\infty) = I_s \left(\frac{R_s R}{R_s + R} \right)$$

This circuit can now be solved given any values of I_s , R_s , L , C , and R .

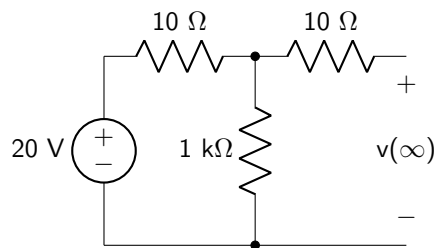
Example: Constant input non-homogeneous general second order circuit in numeric form

Calculate $v(t)$ for the circuit below. The switch closes at a time of 0 seconds.



The initial voltage drop will be zero due to the lack of connection to the voltage source. The voltage dropped over the capacitor will be continuous at $t = 0$.

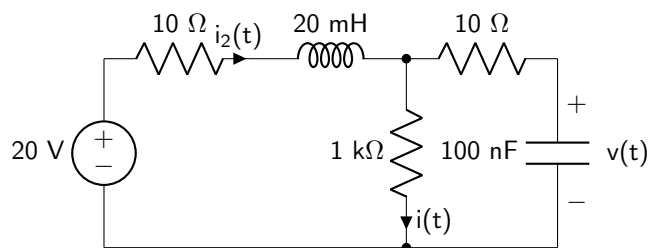
The final value of the voltage can be determined by analyzing the circuit in the steady-state condition. The inductor can be replaced with a short and the capacitor can be replaced with an open.



Use a voltage divider to calculate $v(\infty)$, which is equal to the value of the particular solution.

$$v(\infty) = 20 \text{ V} \left(\frac{1000 \Omega}{1010 \Omega} \right) = 19.80 \text{ V}$$

Analyze the circuit with the switch closed to determine the second order equation. Some other intermediate terms ($i(t)$ and $i_2(t)$) will be introduced to assist in the analysis.



Perform KVL around the loop consisting of the voltage source, both 10Ω resistors, the inductor, and the capacitor. All quantities will be in terms of Ω , V, A, and H.

$$20 = 10i_2(t) + 0.02 \frac{di_2(t)}{dt} + (1 \times 10^{-6}) \frac{dv(t)}{dt} + v(t)$$

Perform KCL at the node connecting the inductor, the $1 \text{ k}\Omega$ resistor, and the 10Ω resistor. (This equation

will be used below to calculate the value of the initial derivative of the voltage.)

$$i_2(t) = i(t) + (100 \times 10^{-9}) \frac{dv(t)}{dt}$$

One last equation is required. Perform KVL around the loop consisting of the 1 k Ω resistor, the 10 Ω resistor, and the capacitor. Divide by 1 k Ω .

$$i(t) = (1 \times 10^{-9}) \frac{dv(t)}{dt} + 0.001v(t)$$

There are now three equations and three unknowns. Solve each equation to find a second order equation in terms of $v(t)$. Then, ensure that the highest order term has a coefficient of one.

$$9.9 \times 10^9 = \frac{d^2v(t)}{dt^2} + 10896.04 \frac{dv(t)}{dt} + (5 \times 10^8) v(t)$$

Calculate alpha and omega.

$$\alpha = \frac{10896.04}{2} = 5448.02 \text{ Np/s}$$

$$\omega_0 = \sqrt{5 \times 10^8} = 22360.68 \text{ rad/s}$$

This circuit is underdamped.

Calculate the initial derivative of the voltage. Because $i(t)$ and $i_2(t)$ will both be zero at $t = 0$ s (and are continuous when the switch closes), $v'(0)$ will be zero.

Use equation 7.13 to calculate the oscillation frequency of the circuit.

$$\beta = \sqrt{22360.68^2 - 5448.02^2} = 21686.84 \text{ rad/s}$$

Use equations 7.39 and 7.40 to calculate the coefficients of the equation.

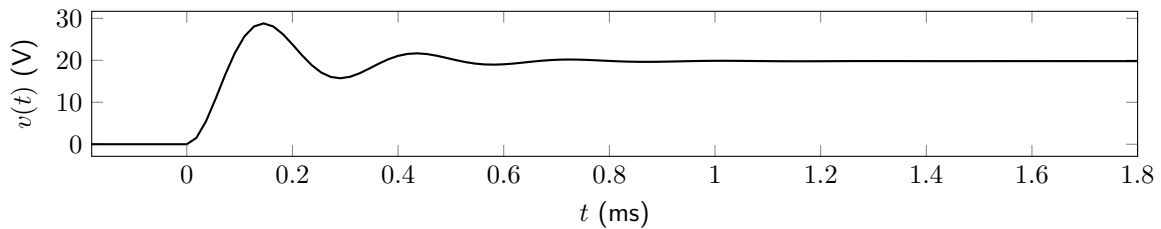
$$B_1 = -19.80 \text{ V}$$

$$B_2 = \frac{-(5448.02)(19.80)}{21686.84} = -4.97 \text{ V}$$

Plug all of these quantities into equation 7.33 to obtain the equation for $i(t)$.

$$v(t) = (19.80 \text{ V} + e^{-5448.02t} [-19.80 \text{ V} \cos(21686.84t) - 4.97 \text{ V} \sin(21686.84t)]) u(t)$$

The current is plotted below as a function of time.



7.5.2 Variable Input Non-Homogeneous Second Order Circuits

It is possible for a second order circuit to be connected to a variable input source (rather than a DC source) in the final circuit configuration. In this case, the particular solution will no longer be equal to that given in equation 7.34. The particular solution for commonly used sources is given in table 7.1.

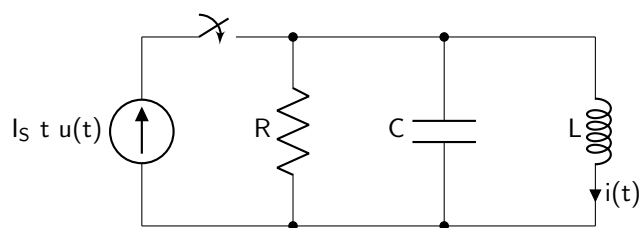
Source Type	Equation	Particular Solution	Notes
Polynomial	$A_2t^2 + A_1t + A_0$	$K_2t^2 + K_1t + K_0$	If 0 is not a root of the characteristic equation.
		$K_2t^3 + K_1t^2 + K_0t$	If 0 is a single root of the characteristic equation.
		$K_2t^4 + K_1t^3 + K_0t^2$	If 0 is a double root of the characteristic equation.
Exponential	Ae^{-at}	Ke^{-at}	If a is not a root of the characteristic equation.
		Kte^{-at}	If a is a single root of the characteristic equation.
		Kt^2e^{-at}	If a is a double root of the characteristic equation.
Cosine	$A \cos \omega t$	$K_1 \cos \omega t + K_2 \sin \omega t$	
Sine	$A \sin \omega t$	$K_1 \cos \omega t + K_2 \sin \omega t$	

Table 7.1: Particular solutions for commonly used variable sources.

To solve for the coefficients of the solution, solve the equation for $v(0)$ and $v'(0)$ and use linear algebra to determine the coefficients. These calculations will not be derived for all variable input sources in this textbook.

Example: Variable input non-homogeneous general second order circuit in symbolic form

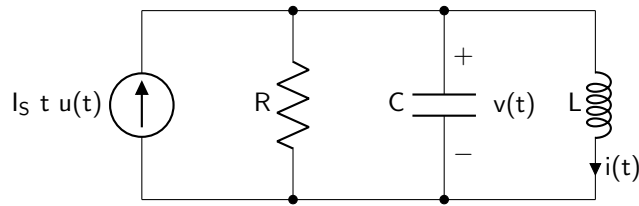
Calculate $i(t)$ for the circuit below. The switch closes at a time of 0 seconds.



The initial current flow will be zero, due to the lack of connection to the voltage source. It is no longer relevant to discuss steady-state values as the input is a ramp function and will not lead to a constant output

value.

Analyze the circuit with the switch closed to determine the second order equation. The voltage drop over the inductor $v(t)$ will also be used to assist in the analysis.



Perform KCL.

$$I_S t = \frac{v(t)}{R} + i(t) + C \frac{dv(t)}{dt}$$

Use equation 6.4 to replace $v(t)$ terms with $i(t)$ terms. Divide all terms until the second order term has a coefficient of one. Note that this is simply a non-homogeneous parallel RLC circuit, so the equation looks very similar to equation 7.23.

$$\frac{I_S}{LC} t = \frac{d^2 i(t)}{dt^2} + \frac{1}{RC} \frac{di(t)}{dt} + \frac{1}{LC} i(t)$$

The particular form of the solution will need to be obtained. Consult table 7.1

$$i_p(t) = K_1 t + K_2$$

Plug the particular solution into the second order equation to calculate the values of the coefficients K_1 and K_2 .

$$\frac{I_S}{LC} t = \frac{d^2}{dt^2} [K_1 t + K_2] + \frac{1}{RC} \frac{d}{dt} [K_1 t + K_2] + \frac{1}{LC} [K_1 t + K_2]$$

Perform all of the derivatives. Then, combine all of the constant terms together in one equation and all of the t terms together in another equation.

$$0 = \frac{1}{RC} K_1 + K_2$$

$$\frac{I_S}{LC} = \frac{1}{LC} K_1$$

These two equations and two unknowns can be put into a matrix to solve for the coefficients.

$$K_1 = I_S$$

$$K_2 = -\frac{I_S}{RC}$$

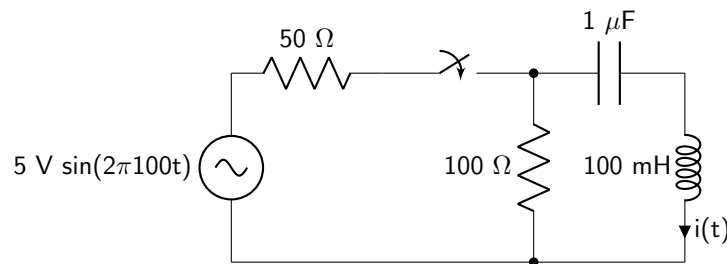
The particular solution is now known.

$$i_p(t) = I_S t - \frac{I_S}{RC}$$

At this point, it is not possible to go any farther in the calculations due to the symbolic form of the solution. The circuit could be overdamped, underdamped, or critically damped. The general solution of the equation cannot be solved without specific component values.

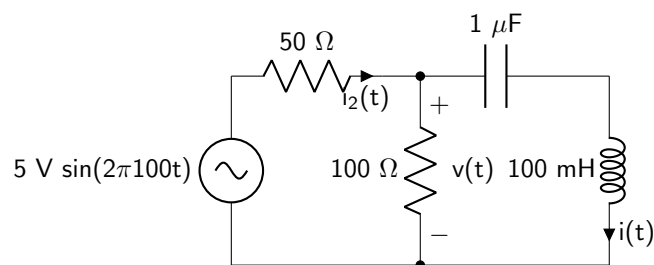
Example: Variable input non-homogeneous general second order circuit in numeric form

Calculate $i(t)$ for the circuit below. The switch closes at a time of 0 seconds.



The initial current flow will be zero, due to the lack of connection to the voltage source. This current flow will be continuous at time $t = 0$. It is no longer relevant to discuss steady-state values as the input is a sinusoidal function and will not lead to a constant output value.

Analyze the circuit with the switch closed to determine the second order equation. Some other intermediate terms ($v(t)$ and $i_2(t)$) will be introduced to assist in the analysis.



Perform KVL around the loop consisting of the voltage source, the 50Ω resistors, the capacitor, and the

inductor. All quantities will be in terms of Ω , V, A, and H.

$$5 \sin(2\pi 100t) = 50i_2(t) + (1 \times 10^6) \int_{-\infty}^t i(\tau) d\tau + 0.1 \frac{di(t)}{dt}$$

Perform KCL at the node connecting both resistors and the capacitor.

$$i_2(t) = 0.01v(t) + i(t)$$

This latter equation can be plugged into the former to obtain an equation only in terms of $v(t)$ and $i(t)$.

$$5 \sin(2\pi 100t) = 0.5v(t) + 50i(t) + (1 \times 10^6) \int_{-\infty}^t i(\tau) d\tau + 0.1 \frac{di(t)}{dt}$$

One last equation is required. Perform KVL around the loop consisting of the 100Ω resistor, the inductor, and the capacitor.

$$v(t) = (1 \times 10^6) \int_{-\infty}^t i(\tau) d\tau + 0.1 \frac{di(t)}{dt}$$

Plug the latter expression into the former to obtain a second order equation that is only in terms of $i(t)$.

$$6666.67\pi \cos(2\pi 100t) = \frac{d^2i(t)}{dt^2} + 333.33 \frac{di(t)}{dt} + (1 \times 10^7) i(t)$$

Calculate alpha and omega.

$$\alpha = \frac{333.33}{2} = 166.67 \text{ Np/s}$$

$$\omega_0 = \sqrt{1 \times 10^7} = 3162.28 \text{ rad/s}$$

This circuit is underdamped.

Use equation 7.13 to calculate the oscillation frequency of the circuit.

$$\beta = \sqrt{3162.28^2 - 166.67^2} = 3157.88 \text{ rad/s}$$

Calculate the value of the particular solution. Because the input is a sine wave, the particular solution will take the form given below.

$$i_p(t) = K_1 \cos(2\pi 100t) + K_2 \sin(2\pi 100t)$$

Plug the particular solution into the second order equation as shown to calculate the values of the

coefficients K_1 and K_2 .

$$\begin{aligned} 6666.67\pi \cos(2\pi 100t) &= \frac{d^2}{dt^2} [K_1 \cos(2\pi 100t) + K_2 \sin(2\pi 100t)] \\ &+ 333.33 \frac{d}{dt} [K_1 \cos(2\pi 100t) + K_2 \sin(2\pi 100t)] \\ &+ (1 \times 10^7) [K_1 \cos(2\pi 100t) + K_2 \sin(2\pi 100t)] \end{aligned}$$

Perform all of the derivatives. Then, combine all of the sine terms together in one equation and all of the cosine terms together in another equation.

$$\begin{aligned} 0 &= -40000\pi^2 K_2 - 66666.67\pi K_1 + (1 \times 10^7) K_2 \\ 6666.67\pi &= -40000\pi^2 K_1 + 66666.67\pi K_2 + (1 \times 10^7) K_1 \end{aligned}$$

These two equations and two unknowns can be put into a matrix to solve for the coefficients.

$$K_1 = 2.18 \times 10^{-3} \text{ A}$$

$$K_2 = 4.75 \times 10^{-5} \text{ A}$$

The particular solution is now known.

$$i_p(t) = (2.18 \times 10^{-3}) \cos(2\pi 100t) + (4.75 \times 10^{-5}) \sin(2\pi 100t)$$

Because the equation is underdamped, equation 7.33 can be used to find an equation for $i(t)$.

$$\begin{aligned} i(t) &= (2.18 \times 10^{-3}) \cos(2\pi 100t) + (4.75 \times 10^{-5}) \sin(2\pi 100t) \\ &+ e^{-166.67t} [B_1 \cos(3157.88t) + B_2 \sin(3157.88t)] \end{aligned}$$

The last step is to calculate the values of coefficients B_1 and B_2 . This is where the initial conditions $i(0)$ and $i'(0^+)$ are used. Both of these values are zero. The first equation to be used to calculate the coefficients is to solve $i(t)$ for $t = 0$ s.

$$i(0) = 0 = (2.18 \times 10^{-3}) + B_1$$

The second equation to be used is to take the first derivative of $i(t)$ and solve for when $t = 0$ s.

$$i'(0^+) = 0 = 3157.88B_2 - 166.67B_1 + 0.0298451$$

Both coefficients can now be calculated.

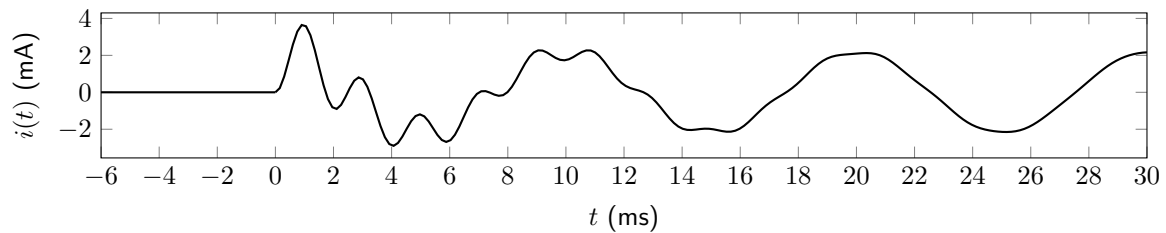
$$B_1 = -2.18 \times 10^{-3} \text{ A}$$

$$B_2 = -1.25 \times 10^{-4} \text{ A}$$

Put everything together to find an equation for $i(t)$.

$$i(t) = [(2.18 \times 10^{-3}) \text{ A} \cos(2\pi 100t) + (4.75 \times 10^{-5}) \text{ A} \sin(2\pi 100t) \\ + e^{-166.67t} [(-2.18 \times 10^{-3}) \text{ A} \cos(3157.88t) - (1.25 \times 10^{-4}) \text{ A} \sin(3157.88t)]] u(t)$$

The current is plotted below as a function of time.



Example Problems

Homogeneous Second Order Circuits

1. Calculate an expression for $v(t)$ given the circuit shown in figure 7.7. The switch opens at a time of zero seconds.

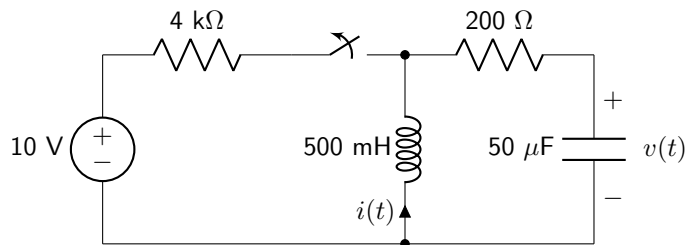


Figure 7.7: Circuit diagram for homogeneous second order circuits question 1.

2. Calculate an expression for $i(t)$ given the circuit shown in figure 7.8. The switch opens at a time of zero seconds.

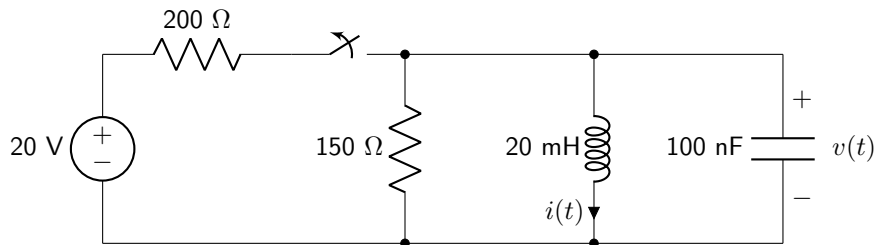


Figure 7.8: Circuit diagram for homogeneous second order circuits question 2.

3. Calculate an expression for $v(t)$ given the circuit shown in figure 7.9. The switch opens at a time of zero seconds.

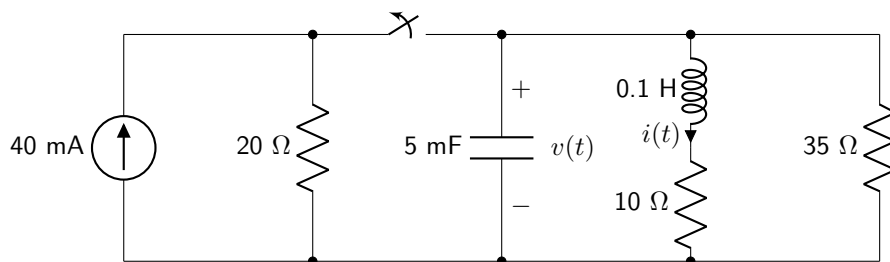


Figure 7.9: Circuit diagram for homogeneous second order circuits question 3.

4. Calculate an expression for $v(t)$ given the circuit shown in figure 7.10. The switch moves from position a to position b at a time of zero seconds.

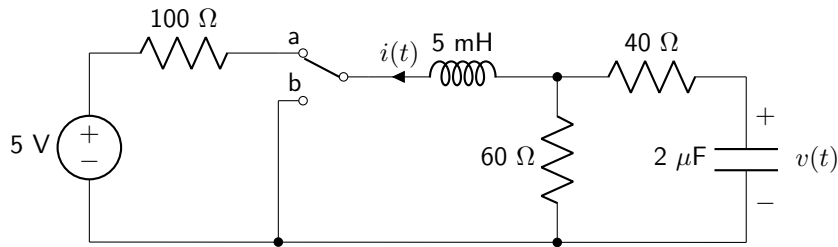


Figure 7.10: Circuit diagram for homogeneous second order circuits question 4.

5. Derive a second order differential equation in terms of $i(t)$ given the circuit shown in figure 7.11. The switch opens at a time of zero seconds.

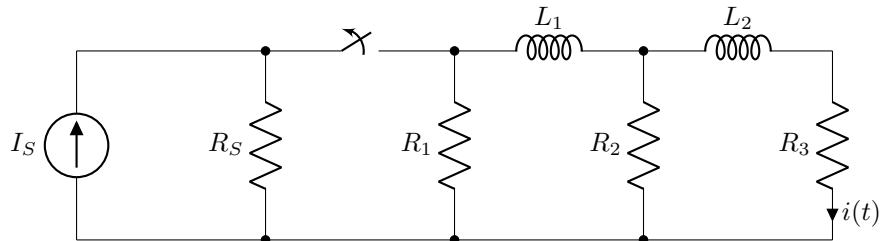


Figure 7.11: Circuit diagram for homogeneous second order circuits question 5.

Non-Homogeneous Second Order Circuits

6. Calculate an expression for $v(t)$ given the circuit shown in figure 7.12. The switch moves from position a to b at a time of zero seconds.

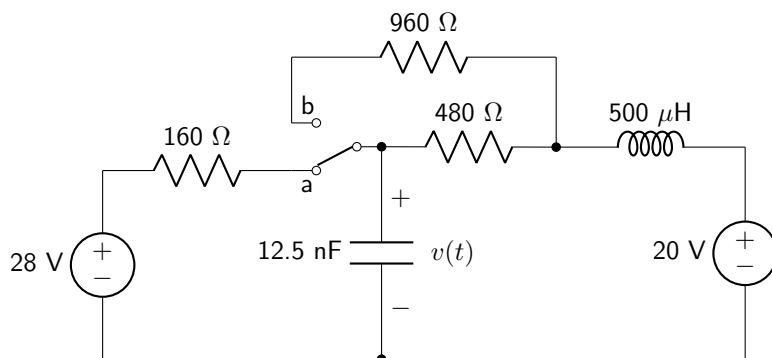


Figure 7.12: Circuit diagram for non-homogeneous second order circuits question 6.

7. Calculate an expression for $i(t)$ given the circuit shown in figure 7.13. The switch closes at a time of zero seconds.

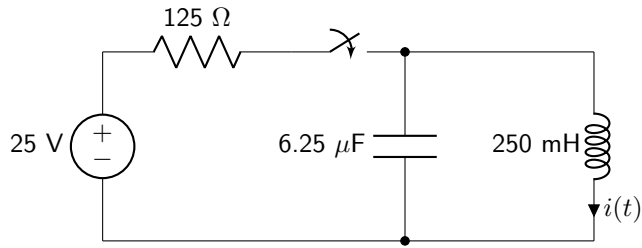


Figure 7.13: Circuit diagram for non-homogeneous second order circuits question 7.

8. Calculate an expression for $v(t)$ given the circuit shown in figure 7.14. The switch moves from position a to b at a time of zero seconds.

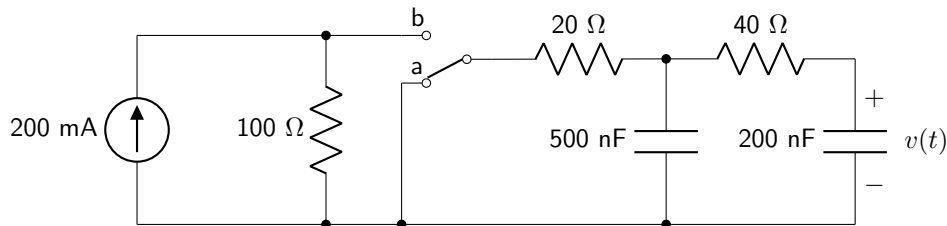


Figure 7.14: Circuit diagram for non-homogeneous second order circuits question 8.

9. Calculate an expression for $i(t)$ given the circuit shown in figure 7.15. Switch S1 opens at a time of zero seconds and switch S2 closes at a time of zero seconds.

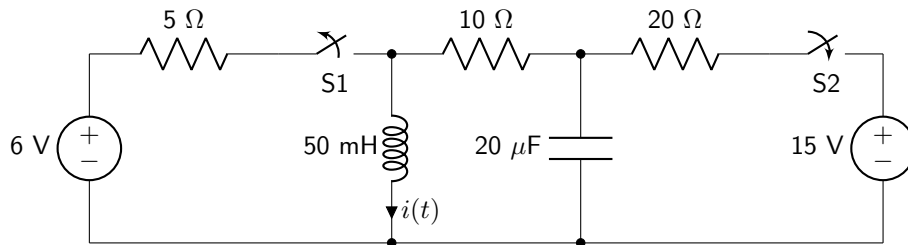


Figure 7.15: Circuit diagram for non-homogeneous second order circuits question 9.

10. Calculate an expression for $v(t)$ given the circuit shown in figure 7.16. The switch closes at a time of zero seconds.

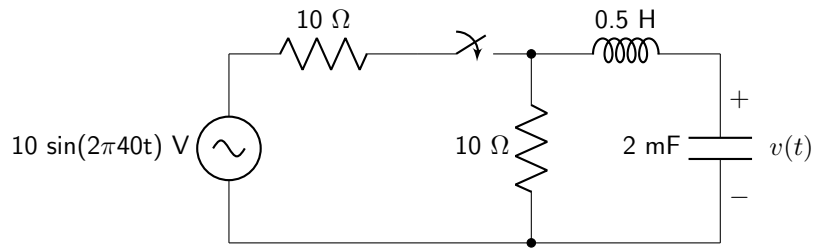


Figure 7.16: Circuit diagram for non-homogeneous second order circuits question 10.

8 Phasors and Impedances

A phasor is a mathematical construct that uses complex numbers (and Euler's equation) to analyze the AC steady-state behavior of circuits. While it might initially seem scary to use complex numbers to analyze circuits, it removes all of the calculus from the analysis and replaces it with addition, subtraction, multiplication, and division.

Euler's equation (shown in equation 8.1) provides a relationship between the argument of a cosine wave and exponent of the base of the natural logarithm. The variables in the expression correspond to the amplitude of the wave (A), the phase of the wave (ϕ), the frequency of the wave (ω), and the square root of negative one (j).

$$A \cos(\omega t + \phi) = A e^{j(\omega t + \phi)} \quad (8.1)$$

When a circuit contains a sinusoidal source, all of the voltages and currents of the circuit will have a sinusoidal behavior. None of the circuit elements discussed in this book are capable of changing the frequency of the response. Therefore, the only truly important properties of each sinusoid are the amplitude and phase. (The frequency is important, but does not need to be included in calculations because it is constant throughout a circuit, and because the amplitude and phase of sinusoidal quantities will not be affected by the frequency in any calculations.) The review of sinusoidal waves in section 1.5.2 of this book should be consulted if there is any question about what is meant by the amplitude or phase of a wave.

Euler's equation reduces to $e^{j\phi}$ when angular frequency is omitted. This phasor quantity can be expressed in either Cartesian form (expressed with real and imaginary components) or polar form (expressed with magnitude and phase). These two phasor forms (and how to convert between the two) are discussed in sections 8.1.1 and 8.1.2.

All of the phasors that are constructed in circuit analysis take the form of a cosine wave with positive amplitude. Any waveforms that are described in terms of a sine wave or that contain a negative amplitude will require conversion before analysis can take place. The angular argument of the phasor is usually expressed in degrees between -180° and $+180^\circ$.

Waveforms with a negative amplitude can be converted to a waveform with a positive amplitude by phase-shifting the wave by either positive or negative 180° . A sine wave can be converted into a positive-amplitude cosine wave by subtracting 90° from the phase. Conversions from non-standard to standard form are shown in equations 8.2–8.4.

$$-\cos(\omega t + \phi) = \cos(\omega t + \phi \pm 180\pi/180) \quad (8.2)$$

$$\sin(\omega t + \phi) = \cos(\omega t + \phi - 90\pi/180) \quad (8.3)$$

$$-\sin(\omega t + \phi) = \cos(\omega t + \phi + 90\pi/180) \quad (8.4)$$

Example: Converting negative amplitude cosine to phasor form

Convert the following expression to standard phasor form.

$$v(t) = -110 \cos(2\pi 60t + 210\pi/180) \text{ V}$$

Use equation 8.2 to convert to standard phasor form. It is desirable to keep the value of the angular component between $\pm 180^\circ$, so it is best to subtract 180° rather than add 180° .

$$\begin{aligned} v(t) &= 110 \cos(2\pi 60t + (210 - 180)\pi/180) \text{ V} \\ &= 110 \cos(2\pi 60t + 30\pi/180) \text{ V} \end{aligned}$$

Example: Converting negative amplitude sine to phasor form

Convert the following expression to standard phasor form.

$$i(t) = -20 \sin(2\pi 60t + 120\pi/180) \text{ A}$$

Use equation 8.4 to convert to standard phasor form. Subtract 360° to keep the angle within $\pm 180^\circ$.

$$\begin{aligned} i(t) &= 20 \cos(2\pi 60t + (120 + 90)\pi/180) \text{ A} \\ &= 20 \cos(2\pi 60t + 210\pi/180) \text{ A} \\ &= 20 \cos(2\pi 60t - 150\pi/180) \text{ A} \end{aligned}$$

8.1 Phasor Arithmetic

The major benefit of using phasors is the ability to solve complicated circuit analysis by simply adding, subtracting, multiplying, and dividing. No calculus is required. Before discussing how to perform these operations with phasors, a review of Cartesian and polar forms will be conducted.

Cartesian coordinates are used to explain the location of a point on the imaginary axes using two quantities: the real part and the imaginary part. This is similar to (x,y) coordinates; the x-coordinate corresponds to the real part, and the y-coordinate corresponds to the imaginary part. Polar coordinates are used to explain the location of a point using the magnitude of the vector (distance from the origin) and the angle from the positive real axis. These two forms are shown graphically in figure 8.1.

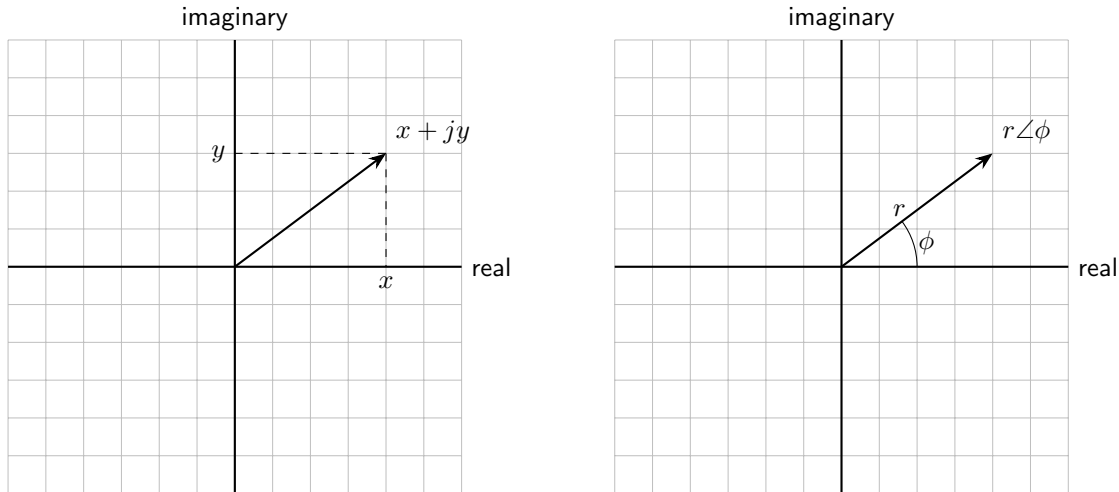


Figure 8.1: A complex number can be described in two ways: Cartesian form (left) and polar form (right).

Note that the symbol j is used to represent the imaginary quantity $\sqrt{-1}$. This is because I and i are reserved in electrical engineering to discuss current. (How can you tell the difference between an electrical engineer and a physicist? Ask them how they represent the square root of negative one!)

8.1.1 Cartesian to Polar Conversion

If the Cartesian form ($x + jy$) of a phasor is known, it is possible to use trigonometry to represent the phasor in polar form as $r \angle \phi$. The magnitude r is found using the Pythagorean theorem, shown in equation 8.5.

$$r = \sqrt{x^2 + y^2} \quad (8.5)$$

To precisely find the angle of a point given Cartesian coordinates, the two-argument arctan function (atan2) should be used, as atan alone is unable to determine in which quadrant a point is located. The atan2 function is defined in equation 8.6.

$$\text{atan2}(y, x) = \begin{cases} \text{atan}\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \text{atan}\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \text{atan}\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases} \quad (8.6)$$

The angle ϕ is found using the atan2 function, shown in equation 8.7.

$$\phi = \text{atan2}(y, x) \quad (8.7)$$

8.1.2 Polar to Cartesian Conversion

If the polar form ($r\angle\phi$) of a phasor is known, it is possible to use trigonometry to express it in Cartesian form as $x + jy$. The real part of the phasor, expressed as x , is found using the cosine of the angle, shown in equation 8.8.

$$x = r \cos \phi \quad (8.8)$$

The imaginary part of the phasor, expressed as y , is found using the sine of the angle, shown in equation 8.9.

$$y = r \sin \phi \quad (8.9)$$

8.1.3 Phasor Addition and Subtraction

Phasor addition and subtraction is simplest to complete when the phasors are in Cartesian form. To add two phasors together, simply add the real parts together to find the real part of the sum and add the imaginary parts together to find the imaginary part of the sum. The solution to $\mathbf{Z}_1 + \mathbf{Z}_2$ is shown in equation 8.10.

$$\mathbf{Z}_1 + \mathbf{Z}_2 = (x_1 + x_2) + j(y_1 + y_2) \quad (8.10)$$

Subtraction is much the same. Remember that subtraction, unlike addition, is not commutative. The solution to $\mathbf{Z}_1 - \mathbf{Z}_2$ is shown in equation 8.11.

$$\mathbf{Z}_1 - \mathbf{Z}_2 = (x_1 - x_2) + j(y_1 - y_2) \quad (8.11)$$

Example: Addition of two sinusoids

Add together $v_1(t)$ and $v_2(t)$, defined below, and express the answer in cosine form.

$$v_1(t) = 10 \text{ V } \cos(2\pi 100t + 30\pi/180)$$

$$v_2(t) = -5 \text{ V } \sin(2\pi 100t - 45\pi/180)$$

Use equation 8.4 to convert $v_2(t)$ to correct form to be converted into a phasor.

$$v_2(t) = 5 \text{ V} \cos(2\pi 100t + 45\pi/180)$$

Both of the terms can be expressed as phasors. Cartesian form will lead to easier addition.

$$\mathbf{V}_1 = 10 \text{ V} \angle 30^\circ = 8.66 + j5 \text{ V}$$

$$\mathbf{V}_2 = 5 \text{ V} \angle 45^\circ = 3.54 + j3.54 \text{ V}$$

Add both of the vectors together.

$$\mathbf{V}_1 + \mathbf{V}_2 = 12.20 + j8.54 \text{ V}$$

Use equations 8.5 and 8.7 to convert to polar form and express as a cosine.

$$v_1(t) + v_2(t) = 14.89 \text{ V} \cos(2\pi 100t + 34.99\pi/180)$$

Example: Subtraction of two sinusoids

Subtract $i_2(t)$ from $i_1(t)$, defined below, and express the answer in cosine form.

$$i_1(t) = 120 \text{ mA} \cos(2\pi 60t + 60\pi/180)$$

$$i_2(t) = 110 \text{ mA} \cos(2\pi 60t - 135\pi/180)$$

Both of the terms can be expressed as phasors. Cartesian form will lead to easier subtraction.

$$\mathbf{I}_1 = 120 \text{ mA} \angle 60^\circ = 60 + j103.92 \text{ mA}$$

$$\mathbf{I}_2 = 110 \text{ mA} \angle -135^\circ = -77.78 - j77.78 \text{ mA}$$

Perform the subtraction.

$$\mathbf{I}_1 - \mathbf{I}_2 = 137.78 + j181.70 \text{ mA}$$

Use equations 8.5 and 8.7 to convert to polar form and express as a cosine.

$$i_1(t) - i_2(t) = 228.04 \text{ mA} \cos(2\pi 60t + 52.83\pi/180)$$

8.1.4 Phasor Multiplication and Division

Phasor multiplication and division is simplest to compute when phasors are in polar form. Recall that polar form ($r\angle\phi$) is shorthand for $re^{j\phi}$. Multiplication and division in exponentials turns into addition and subtraction.

Two phasors Z_1 and Z_2 are multiplied together in equation 8.12.

$$\mathbf{Z}_1\mathbf{Z}_2 = r_1r_2\angle(\phi_1 + \phi_2) \quad (8.12)$$

Two phasors in polar form are divided in equation 8.13. Recall that division, unlike multiplication, is not commutative.

$$\frac{\mathbf{Z}_1}{\mathbf{Z}_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2) \quad (8.13)$$

Example: Multiplication of two sinusoids

Multiply $i(t)$ and $z(t)$, defined below, and express the answer in cosine form.

$$i(t) = 4 \text{ mA} \cos(2\pi 50t + 120\pi/180)$$

$$z(t) = 20 \text{ k}\Omega \cos(2\pi 50t - 90\pi/180)$$

Both of the terms can be expressed as phasors. Polar form will lead to easier multiplication.

$$\mathbf{I} = 4 \text{ mA}\angle 120^\circ$$

$$\mathbf{Z} = 20 \text{ k}\Omega\angle -90^\circ$$

Perform the multiplication.

$$\mathbf{IZ} = 80 \text{ V}\angle 30^\circ$$

It is now possible to express this phasor as a cosine.

$$v(t) = 80 \text{ V} \cos(2\pi 50t + 30\pi/180)$$

Example: Division of two sinusoids

Divide $v(t)$ by $i(t)$, defined below, and express the answer in cosine form.

$$v(t) = 110 \text{ V} \cos(2\pi 60t + 60\pi/180)$$

$$i(t) = 5 \text{ mA} \cos(2\pi 60t + 30\pi/180)$$

Both of the terms can be expressed as phasors. Polar form will lead to easier division.

$$\mathbf{V} = 110 \text{ V} \angle 60^\circ$$

$$\mathbf{I} = 5 \text{ mA} \angle 30^\circ$$

Perform the division.

$$\frac{\mathbf{V}}{\mathbf{I}} = 22 \text{ k}\Omega \angle 30^\circ$$

It is now possible to express this phasor as a cosine.

$$z(t) = 22 \text{ k}\Omega \cos(2\pi 60t + 30\pi/180)$$

8.2 Impedance

Impedance is the equivalent of resistance that exists in AC circuits. It relates to the measure of opposition to current flow based on a given voltage rise or drop. Using time-varying signals, $z(t) = v(t)/i(t)$. Using phasors, the relationship between voltage, current, and impedance is given in equation 8.14. The units of impedance is the same as that of resistance, Ω (ohms).

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \tag{8.14}$$

As impedance is a complex quantity, it can be represented in cartesian form, shown in equation 8.15. The quantity R is the resistance of the circuit element, and X is the reactance of the circuit element.

$$\mathbf{Z} = R + jX \tag{8.15}$$

Impedance can be calculated for each circuit element (resistor, capacitor, and inductor) by using the relationship between voltage and current through that circuit element to find $v(t)$ and $i(t)$.

8.2.1 Resistors

In a resistor, voltage and current are related via Ohm's law (equations 2.21). If the voltage drop over a resistor is given as $V_m \cos(\omega t + \phi)$, then Ohm's law states that the current flowing through the resistor will be equal to $(V_m/R) \cos(\omega t + \phi)$. The conversion of $v(t)$ and $i(t)$ to phasor form, as well as the calculation of the impedance of a resistor, is shown in equation 8.16 and derived below.

$$\mathbf{Z} = R \quad (8.16)$$

DERIVATION

$$\begin{aligned} \mathbf{V} &= V_m \angle \phi \\ \mathbf{I} &= \frac{V_m}{R} \angle \phi \\ \mathbf{Z} &= \frac{\mathbf{V}}{\mathbf{I}} \\ &= \frac{V_m \angle \phi}{(V_m/R) \angle \phi} \\ &= R \end{aligned}$$

The impedance of a resistor is therefore a real quantity and is equal to the resistance. There is no change in the way we express resistance in AC circuits as compared to DC circuits.

8.2.2 Capacitors

In a capacitor, the relationship between voltage and current is given by equation 5.5. If the voltage drop over a capacitor is given as $V_m \cos(\omega t + \phi)$, then equation 5.5 states that the current flowing through the capacitor will be equal to $\omega C V_m \cos(\omega t + \phi + 90\pi/180)$. The conversion of $v(t)$ and $i(t)$ to phasor form, as well as the calculation of the impedance of a capacitor, is shown in equation 8.17 and derived below.

$$\mathbf{Z} = \frac{-j}{\omega C} \quad (8.17)$$

DERIVATION

$$\begin{aligned}
 \mathbf{V} &= V_m \angle \phi \\
 \mathbf{I} &= \omega C V_m \angle (\phi + 90^\circ) \\
 \mathbf{Z} &= \frac{\mathbf{V}}{\mathbf{I}} \\
 &= \frac{V_m \angle \phi}{\omega C V_m \angle (\phi + 90^\circ)} \\
 &= \frac{1}{\omega C} \angle -90^\circ \\
 &= \frac{-j}{\omega C}
 \end{aligned}$$

The impedance of a capacitor is purely imaginary (from a mathematical point of view) and is frequency dependent. As the frequency of a circuit approaches zero, the impedance of a capacitor approaches infinity. This explains why we were able to analyze capacitors in steady-state DC conditions as open circuits. As the frequency of a circuit approaches infinity, the impedance of a capacitor becomes zero, making it act as a short circuit.

8.2.3 Inductors

In an inductor, the relationship between voltage and current is given by equations 6.4 and 6.5. If the current flowing through an inductor is given as $I_m \cos(\omega t + \phi)$, then equation 6.4 states that the voltage drop over the inductor will be equal to $\omega L I_m \cos(\omega t + \phi + 90\pi/180)$. The conversion of $v(t)$ and $i(t)$ to phasor form, as well as the calculation of the impedance of an inductor, is shown in equation 8.18 and derived below.

$$\mathbf{Z} = j\omega L \quad (8.18)$$

DERIVATION

$$\begin{aligned}
 \mathbf{V} &= \omega L I_m \angle (\phi + 90^\circ) \\
 \mathbf{I} &= I_m \angle \phi \\
 \mathbf{Z} &= \frac{\mathbf{V}}{\mathbf{I}} \\
 &= \frac{\omega L I_m \angle (\phi + 90^\circ)}{I_m \angle \phi} \\
 &= \omega L \angle 90^\circ \\
 &= j\omega L
 \end{aligned}$$

The impedance of an inductor is purely imaginary (from a mathematical point of view) and is frequency

dependent. As the frequency of a circuit approaches zero, the impedance of an inductor also approaches zero. This explains why we were able to analyze inductors in steady-state DC conditions as short circuits. As the frequency of a circuit approaches infinity, the impedance of an inductor also approaches infinity, making it act as an open circuit.

8.2.4 Equivalent Impedance

Impedance takes a lot of confusion out of reducing circuits. Impedances in series always add together, as shown in equation 8.19.

$$\mathbf{Z}_{\text{EQ}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \dots + \mathbf{Z}_n \quad (8.19)$$

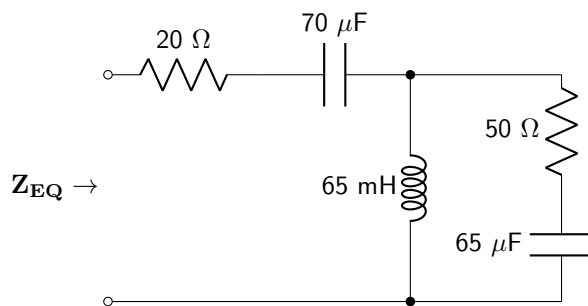
The reciprocal sum is used to calculate parallel impedances, as shown in equation 8.20.

$$\frac{1}{\mathbf{Z}_{\text{EQ}}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \dots + \frac{1}{\mathbf{Z}_n} \quad (8.20)$$

Finding equivalent impedance enables us to reduce to one circuit element the parallel and series combinations of different types of passive elements (resistors, capacitors, and inductors), which we were unable to do in previous chapters.

Example: Equivalent impedance

Calculate the equivalent impedance of the circuit below, and express as individual components. The frequency of the circuit is 100 Hz.



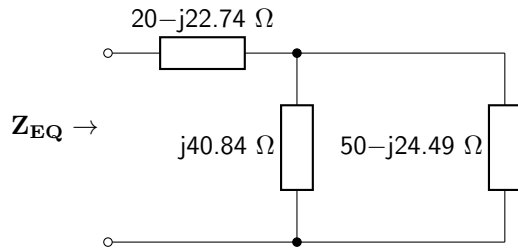
Convert each of the values into impedances. The resistor values will remain unchanged.

$$\mathbf{Z}_{70\mu\text{F}} = -\frac{j}{2\pi 100(70 \times 10^{-6})} = -j22.74 \Omega$$

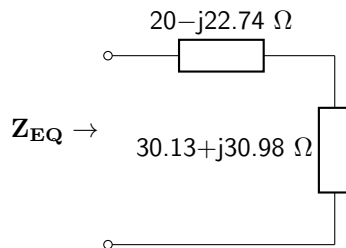
$$\mathbf{Z}_{65\text{mH}} = j2\pi 100(0.065) = j40.94 \Omega$$

$$\mathbf{Z}_{65\mu\text{F}} = -\frac{j}{2\pi 100(65 \times 10^{-6})} = -j24.49 \Omega$$

Re-draw the circuit.

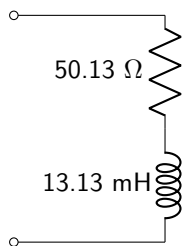


Two of the impedances ($j40.84 \Omega$ and $50 - j24.49 \Omega$) are connected in parallel and can be reduced to an equivalent impedance of $30.13 + j30.98 \Omega$. The circuit can be re-drawn.



The last two impedances can be combined in series to an equivalent impedance of $50.13 + j8.25 \Omega$. This impedance consists of a resistor (50.13Ω) and an inductor. The value of that inductor can be calculated.

$$L = \frac{8.25}{2\pi 100} = 13.13 \text{ mH}$$



8.3 Delta-Wye and Wye-Delta Transforms

The concept of delta-wye and wye-delta transforms was discussed in section 2.3. These transforms can be used to find the equivalent impedance of circuit elements that are neither in parallel nor in series with each other.

A delta circuit arrangement of impedances is shown in figure 8.2.

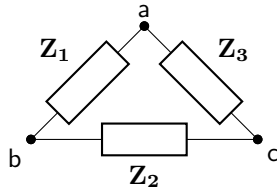


Figure 8.2: A delta circuit with impedance rather than resistance.

The impedance between any two nodes in a delta circuit are parallel and series combinations of those impedances, shown in equations 8.21–8.23.

$$\mathbf{Z}_{ab} = \mathbf{Z}_1 // (\mathbf{Z}_2 + \mathbf{Z}_3) \quad (8.21)$$

$$\mathbf{Z}_{bc} = \mathbf{Z}_2 // (\mathbf{Z}_1 + \mathbf{Z}_3) \quad (8.22)$$

$$\mathbf{Z}_{ac} = \mathbf{Z}_3 // (\mathbf{Z}_1 + \mathbf{Z}_2) \quad (8.23)$$

A wye circuit arrangement of impedances is shown in figure 8.3.

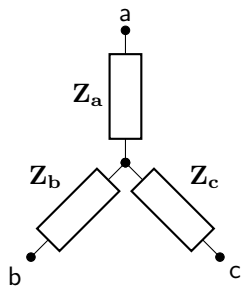


Figure 8.3: A wye circuit with impedance rather than resistance.

The impedance between any two nodes in a wye circuit are series combinations of those impedances, shown in equations 8.24–8.26.

$$\mathbf{Z}_{ab} = \mathbf{Z}_a + \mathbf{Z}_b \quad (8.24)$$

$$\mathbf{Z}_{bc} = \mathbf{Z}_b + \mathbf{Z}_c \quad (8.25)$$

$$\mathbf{Z}_{ac} = \mathbf{Z}_a + \mathbf{Z}_c \quad (8.26)$$

A delta-wye transform modifies a delta arrangement of impedances (figure 8.2) and turns it into a wye arrangement (figure 8.3). The equivalent impedance between any two nodes of the wye circuit must be identical to the corresponding equivalent impedance of the delta circuit in order to make this transformation valid. Equation 8.21 is set equal to equation 8.24; equation 8.22 is set equal to equation 8.25; and equation 8.23 is set equal to equation 8.26. The quantities \mathbf{Z}_a , \mathbf{Z}_b , and \mathbf{Z}_c are solved for. (The algebraic details of these derivations will not be shown in this book.) The results are given in equations 8.27–8.29. These

equations govern the delta-wye transform.

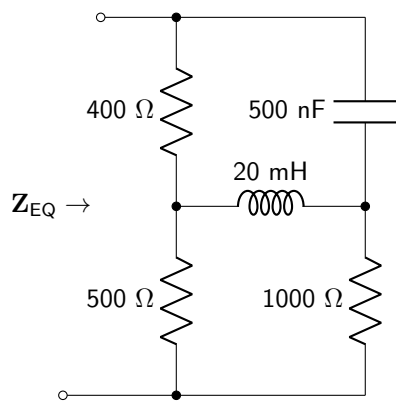
$$\mathbf{Z}_a = \frac{\mathbf{Z}_1 \mathbf{Z}_3}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \quad (8.27)$$

$$\mathbf{Z}_b = \frac{\mathbf{Z}_1 \mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \quad (8.28)$$

$$\mathbf{Z}_c = \frac{\mathbf{Z}_2 \mathbf{Z}_3}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \quad (8.29)$$

Example: Using a delta-wye transformation to calculate equivalent impedance

Calculate the equivalent impedance of the circuit as shown, and express as individual components. The frequency of the circuit is 1 kHz.

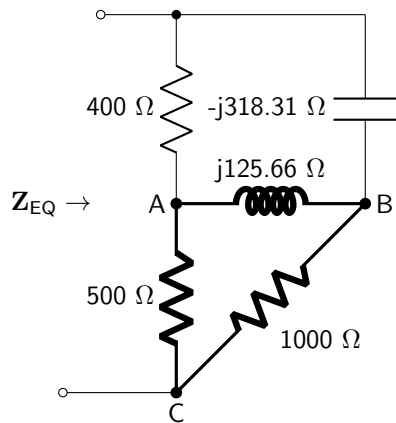


None of these elements shares a branch, therefore none are in series. None of these elements shares two nodes, therefore none are in parallel. A delta-wye transform is required. (Note that a wye-delta transform could be accomplished as well.) Calculate the impedance of the capacitor and the inductor.

$$\mathbf{Z}_L = j2\pi 1000(0.02) = j125.66 \, \Omega$$

$$\mathbf{Z}_C = -\frac{j}{2\pi 1000(500 \times 10^{-9})} = -j318.31 \, \Omega$$

The circuit has been re-drawn. The impedances in the delta-arrangement are highlighted in thick outlines. Each of the nodes is labeled.

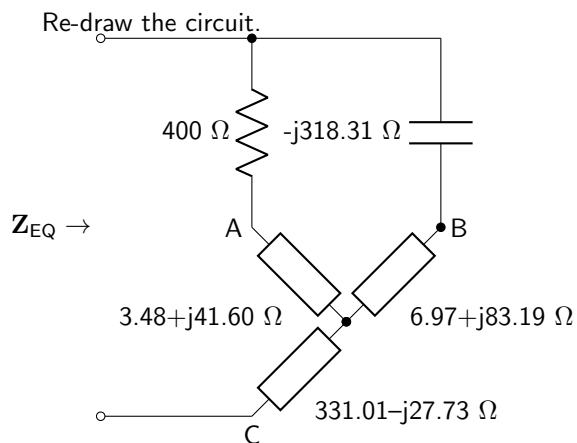


Use equations 8.27–8.29 to solve for \mathbf{Z}_A , \mathbf{Z}_B , and \mathbf{Z}_C .

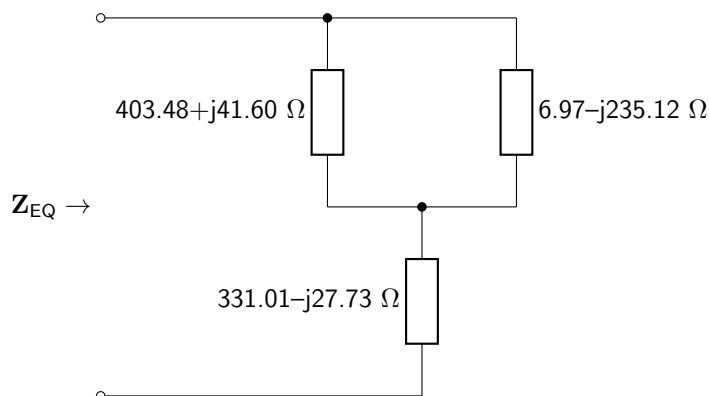
$$\mathbf{Z}_A = \frac{(j125.66 \Omega)(1000 \Omega)}{(j125.66 \Omega) + (1000 \Omega) + (500 \Omega)} = 3.48 + j41.60 \Omega$$

$$\mathbf{Z}_B = \frac{(j125.66 \Omega)(500 \Omega)}{(j125.66 \Omega) + (1000 \Omega) + (500 \Omega)} = 6.97 + j83.19 \Omega$$

$$\mathbf{Z}_C = \frac{(1000 \Omega)(500 \Omega)}{(j125.66 \Omega) + (1000 \Omega) + (500 \Omega)} = 331.01 - j27.73 \Omega$$



Combine all series impedances and re-draw.



Calculate the equivalent impedances of the two upper circuit elements in parallel with each other.

$$\mathbf{Z} = \frac{(403.48 + j41.60 \Omega)(331.01 - j27.73 \Omega)}{(403.48 + j41.60 \Omega) + (331.01 - j27.73 \Omega)} = 113.98 - j176.68 \Omega$$

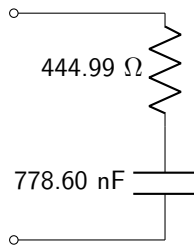
The overall equivalent impedance is equal to the above impedance added to $331.01 - j27.73 \Omega$.

$$\mathbf{Z}_{\text{EQ}} = 444.99 - j204.41 \Omega$$

This impedance is consistent of a resistor in series with a capacitor. The capacitor value is calculated below.

$$C = \frac{1}{2\pi 1000(204.41)} = 778.60 \text{ nF}$$

The circuit can be re-drawn one last time.



A wye-delta transform modifies a wye arrangement of impedances (figure 8.3) and turns it into a delta arrangement (figure 8.2). Just as with the delta-wye transforms, the circuits must be equivalent to each other. Equation 8.21 is set equal to equation 8.24; equation 8.22 is set equal to equation 8.25; and equation 8.23 is set equal to equation 8.26. The quantities \mathbf{Z}_1 , \mathbf{Z}_2 , and \mathbf{Z}_3 are solved for. (The algebraic details of these derivations will not be shown in this book.) The results are given in equation 8.30– 8.32. These equations govern the wye-delta transform.

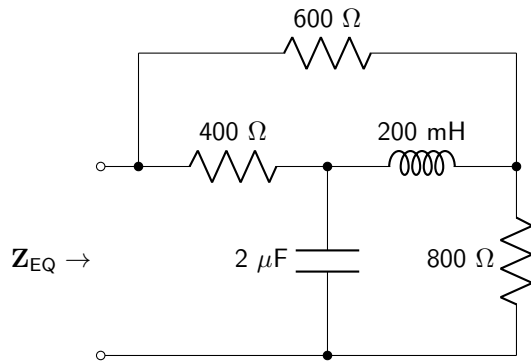
$$\mathbf{Z}_1 = \frac{\mathbf{Z}_a\mathbf{Z}_b + \mathbf{Z}_b\mathbf{Z}_c + \mathbf{Z}_a\mathbf{Z}_c}{\mathbf{Z}_c} \quad (8.30)$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_a\mathbf{Z}_b + \mathbf{Z}_b\mathbf{Z}_c + \mathbf{Z}_a\mathbf{Z}_c}{\mathbf{Z}_a} \quad (8.31)$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b + \mathbf{Z}_b\mathbf{Z}_c + \mathbf{Z}_a\mathbf{Z}_c}{\mathbf{Z}_b} \quad (8.32)$$

Example: Using a wye-delta transformation to calculate equivalent impedance

Calculate the equivalent impedance of the circuit as shown, and express as individual components. The frequency of the circuit is 200 Hz.

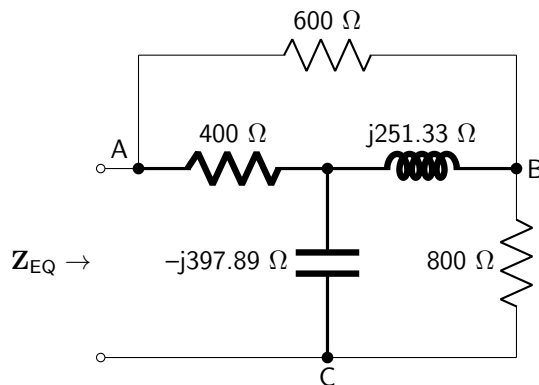


None of these elements shares a branch, therefore none are in series. None of these elements shares two nodes, therefore none are in parallel. A wye-delta transform is required. (Note that a delta-wye transform could be accomplished as well, but it is probably more obvious to do a wye-delta transform.) Calculate the impedance of the capacitor and the inductor.

$$\mathbf{Z}_L = j2\pi 200(0.2) = j251.33 \Omega$$

$$\mathbf{Z}_C = -\frac{j}{2\pi 200(2 \times 10^{-6})} = -j397.89 \Omega$$

The circuit has been re-drawn. The impedances in the wye-arrangement are highlighted in thick outlines. Each of the nodes is labeled.



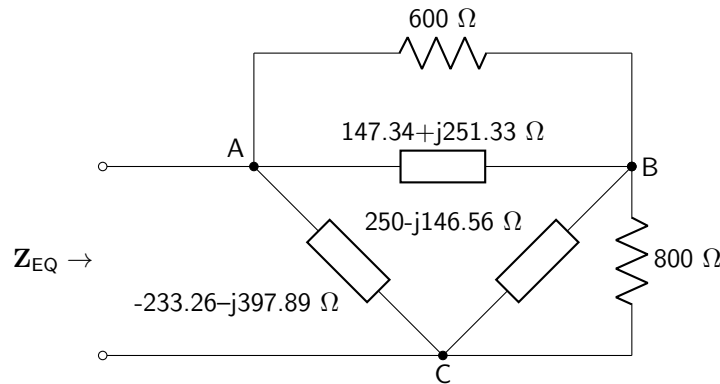
Use equations 8.30–8.32 to solve for \mathbf{Z}_1 , \mathbf{Z}_2 , and \mathbf{Z}_3 .

$$\mathbf{Z}_1 = \frac{100000 - j58623.98 \Omega^2}{-j397.89 \Omega} = 147.34 + j251.33 \Omega$$

$$\mathbf{Z}_2 = \frac{100000 - j58623.98 \Omega^2}{400 \Omega} = 250 - j146.56 \Omega$$

$$\mathbf{Z}_3 = \frac{100000 - j58623.98 \Omega^2}{j251.33 \Omega} = -233.26 - j397.89 \Omega$$

Re-draw the circuit.



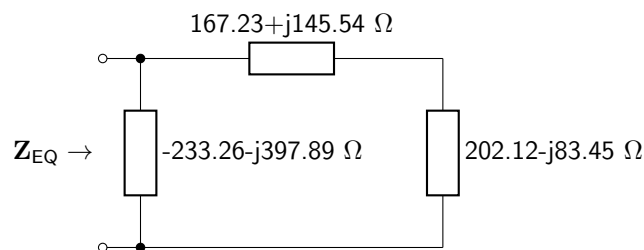
Calculate the equivalent impedances of the 600Ω resistor in parallel with the $147.34 + j251.33 \Omega$ impedance.

$$Z = \frac{(600 \Omega)(147.34 + j251.33 \Omega)}{(600 \Omega) + (147.34 + j251.33 \Omega)} = 167.23 + j145.54 \Omega$$

Calculate the equivalent impedances of the 800Ω resistor in parallel with the $250 - j146.56 \Omega$ impedance.

$$Z = \frac{(800 \Omega)(250 - j146.56 \Omega)}{(800 \Omega) + (250 - j146.56 \Omega)} = 202.12 - j83.45 \Omega$$

Re-draw the circuit.



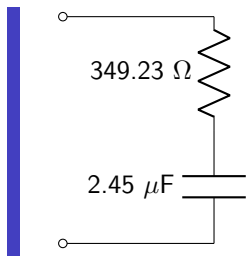
The equivalent impedance is calculated below.

$$\begin{aligned} Z_{EQ} &= \frac{(167.23 + j145.54 \Omega + 202.12 - j83.45 \Omega)(-233.26 - j397.89 \Omega)}{(167.23 + j145.54 \Omega) + (202.12 - j83.45 \Omega) + (-233.26 - j397.89 \Omega)} \\ &= 349.23 - j324.55 \Omega \end{aligned}$$

This impedance is consistent of a resistor in series with a capacitor. The capacitor value is calculated below.

$$C = \frac{1}{2\pi 200(324.55)} = 2.45 \mu\text{F}$$

The circuit can be re-drawn one last time.



Example Problems

Phasor Arithmetic

1. Convert $-200 - j100$ to polar form.
2. Convert $5\angle -120^\circ$ to Cartesian form.
3. Calculate the value of $(10 + j20)(-30 + j50) + (-15 - j40)$ and express the answer in both Cartesian and polar forms.
4. Calculate the value of $(10\angle 60^\circ - 4\angle -140^\circ)(20\angle 20^\circ)$ and express the answer in both Cartesian and polar forms.
5. Calculate the value of $(10 + j14)(60\angle -140^\circ) + (38\angle 20^\circ)/(-5 - j18)$ and express the answer in both Cartesian and polar forms.

Impedance and Equivalent Impedance

6. Calculate the impedance of a 25 mH inductor at a frequency of 60 Hz.
7. Calculate the impedance of a 470 nF capacitor at a frequency of 2 kHz.
8. Calculate the equivalent impedance of the circuit shown in figure 8.4. The frequency of operation is 200 Hz.

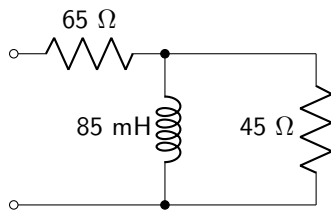


Figure 8.4: Circuit diagram for impedance and equivalent impedance question 8.

9. Calculate the equivalent impedance of the circuit shown in figure 8.5. The frequency of operation is 450 Hz.

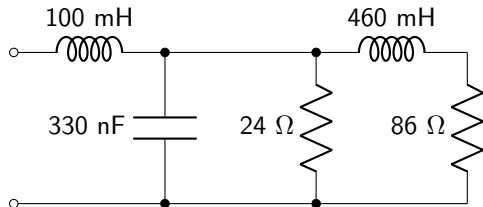


Figure 8.5: Circuit diagram for impedance and equivalent impedance question 9.

10. Calculate the equivalent impedance of the circuit shown in figure 8.6. The frequency of operation is 6 kHz.

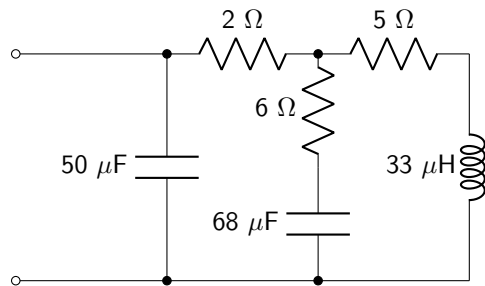


Figure 8.6: Circuit diagram for impedance and equivalent impedance question 10.

Delta-Wye and Wye-Delta Transforms

11. Convert the circuit shown in figure 8.7 to a delta circuit.

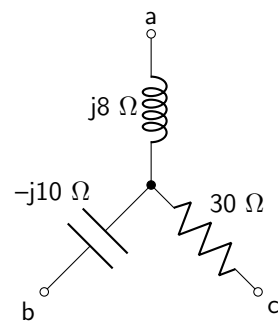


Figure 8.7: Circuit diagram for delta-wye and wye-delta transforms question 11.

12. Convert the circuit shown in figure 8.8 to a wye circuit.

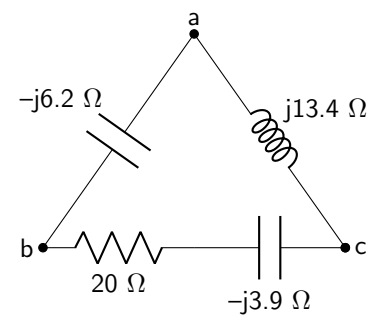


Figure 8.8: Circuit diagram for delta-wye and wye-delta transforms question 12.

13. Calculate the equivalent impedance of the circuit shown in figure 8.9.

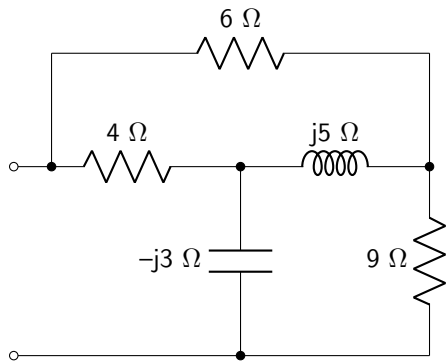


Figure 8.9: Circuit diagram for delta-wye and wye-delta transforms question 13.

14. Calculate the equivalent impedance of the circuit shown in figure 8.10. The frequency of operation is 10 kHz.

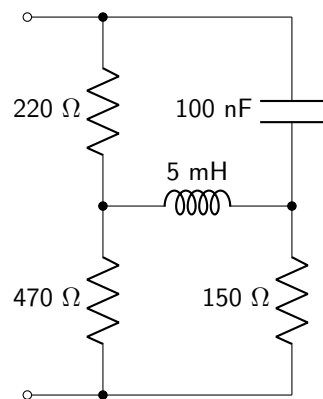


Figure 8.10: Circuit diagram for delta-wye and wye-delta transforms question 14.

15. Calculate the equivalent impedance of the circuit shown in figure 8.11. The frequency of operation is 50 Hz.

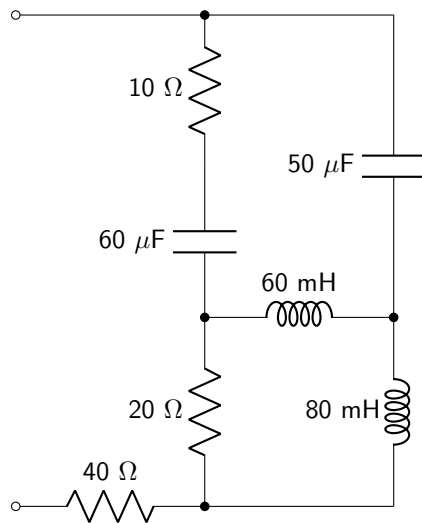


Figure 8.11: Circuit diagram for delta-wye and wye-delta transforms question 15.

9 Analysis of Phasor Transformed Circuits

Now that the concepts of phasors and impedances have been explained, it is possible to use those quantities to analyze the AC characteristics of circuits.

9.1 Complex Circuit Analysis Techniques

All linear circuits can be analyzed using the following circuit analysis techniques. All of them have been covered earlier in this textbook in the context of DC circuits. Phasors and impedances will be used to implement these tools to solve AC circuits. All sources will be converted into phasor quantities, and all passive circuit elements will be converted into impedances.

Because all of these concepts have already been covered, the discussions that follow will not be exhaustive as those that were included in chapters 2 and 3 in this textbook.

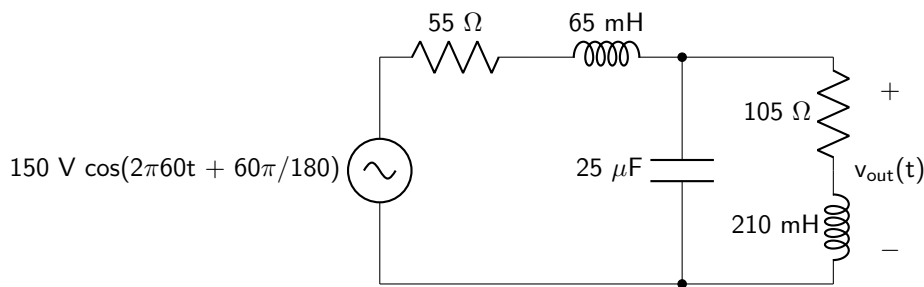
9.1.1 Voltage Divider

The voltage divider uses the complex form of Ohm's law ($\mathbf{V} = \mathbf{IZ}$) to calculate the voltage dropped over any series impedance without having to calculate the current flowing through the series circuit. Given n impedances in series and a source voltage of \mathbf{V}_S , the voltage drop over the k^{th} impedance (where $k < n$) is shown in equation 9.1.

$$\mathbf{V}_k = \mathbf{V}_S \left(\frac{\mathbf{Z}_k}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \dots + \mathbf{Z}_k + \dots + \mathbf{Z}_n} \right) \quad (9.1)$$

Example: Complex voltage divider

Calculate $v_{\text{out}}(t)$ in the circuit below.



Calculate the impedance \mathbf{Z}_1 containing the $55 \text{ } \Omega$ resistor and 65 mH inductor.

$$\mathbf{Z}_1 = 55 + j2\pi 60(0.065) \text{ } \Omega = 55 + j24.50 \text{ } \Omega$$

Calculate the impedance of the capacitor.

$$\mathbf{Z}_C = \frac{-j}{2\pi 60(25 \times 10^{-6})} = -j106.10 \Omega$$

Calculate the impedance of the 105 Ω resistor and 210 mH inductor branch.

$$\mathbf{Z}_{RL} = 105 + j2\pi 60(.210) \Omega = 105 + j79.17 \Omega$$

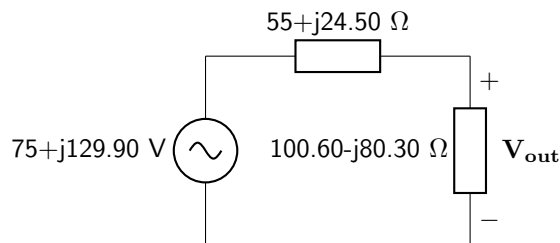
The capacitor is in parallel with the resistor/inductor branch. Calculate the parallel impedance.

$$\mathbf{Z}_2 = \frac{(-j106.10 \Omega)(105 + j79.17 \Omega)}{(-j106.10 \Omega) + (105 + j79.17 \Omega)} = 100.60 - j80.30 \Omega$$

Convert the voltage source into a phasor quantity.

$$\mathbf{V} = 150 \cos 60 + j150 \sin 60 \text{ V} = 75 + j129.90 \text{ V}$$

The circuit can now be re-drawn.



Apply the voltage divider equation to find \mathbf{V}_{out} in phasor form.

$$\begin{aligned} \mathbf{V}_{\text{out}} &= \mathbf{V}_S \left(\frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} \right) \\ &= (75 + j129.90 \text{ V}) \left(\frac{100.60 - j80.30 \Omega}{(55 + j24.50 \Omega) + (100.60 - j80.30 \Omega)} \right) \\ &= 87.98 + j76.83 \text{ V} \\ &= 116.80 \text{ V} \angle 41.13^\circ \end{aligned}$$

Convert the output to a time-varying signal.

$$v_{\text{out}}(t) = 116.80 \text{ V} \cos(2\pi 60t + 41.13\pi/180)$$

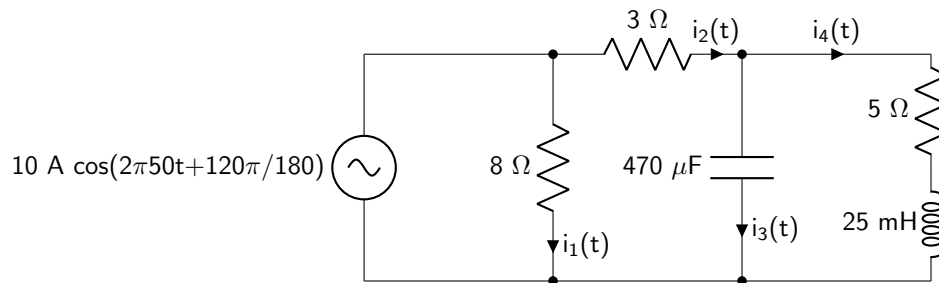
9.1.2 Current Divider

The current divider uses the complex form of Ohm's law to calculate the current flowing through any parallel impedance without having to calculate the voltage dropped over the parallel circuit. Given n impedances in parallel and a source current of \mathbf{I}_S , the current flow through the k^{th} impedance (where $k < n$) is shown in equation 9.2.

$$\mathbf{I}_k = \mathbf{I}_S \left(\frac{\mathbf{Z}_{\text{EQ}}}{\mathbf{Z}_k} \right) \quad (9.2)$$

Example: Complex current divider

Use a current divider to calculate each of the currents as shown.



Calculate the impedances of the inductor and the capacitor.

$$\mathbf{Z}_L = j2\pi 50(0.025) = j7.85 \Omega$$

$$\mathbf{Z}_C = -\frac{j}{2\pi 50(470 \times 10^{-6})} = -j6.77 \Omega$$

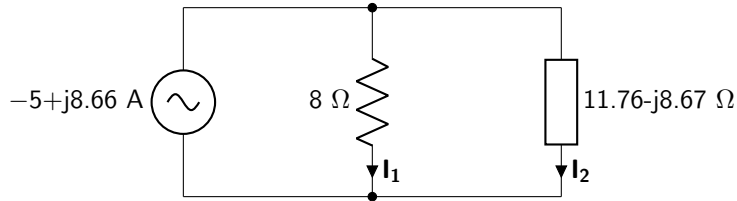
Convert the current source to phasor form.

$$\mathbf{I} = 10 \cos 120^\circ + j10 \sin 120^\circ \text{ A} = -5 + j8.66 \text{ A}$$

Calculate an equivalent impedance consisting of the 3 Ω resistor, the capacitor, the 5 Ω resistor, and the inductor

$$\mathbf{Z} = (5 \Omega) + (j7.85 \Omega) // (-j6.77 \Omega) + 3 \Omega = 11.76 - j8.67 \Omega$$

Re-draw the circuit



Use the current divider rule to calculate \mathbf{I}_1 and \mathbf{I}_2 .

$$\mathbf{I}_1 = (-5 + j8.66 \text{ A}) \frac{(8 \Omega) / (11.76 - j8.67 \Omega)}{8 \Omega} = -2.01 + j6.46 \text{ A}$$

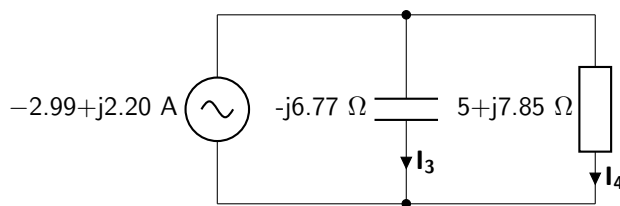
$$\mathbf{I}_2 = (-5 + j8.66 \text{ A}) \frac{(8 \Omega) / (11.76 - j8.67 \Omega)}{11.76 - j8.67 \Omega} = -2.99 + j2.20 \text{ A}$$

Convert both of these values into time-varying signals.

$$i_1(t) = 6.77 \text{ A} \cos(2\pi 50t + 107.30\pi/180)$$

$$i_2(t) = 3.71 \text{ A} \cos(2\pi 50t + 143.68\pi/180)$$

Re-draw the circuit. Current $i_2(t)$ can be used to represent a source current supplying the capacitor, 5 Ω resistor and inductor.



Use the current divider rule to calculate \mathbf{I}_3 and \mathbf{I}_4 .

$$\mathbf{I}_3 = (-2.99 + j2.20 \text{ A}) \frac{(-j6.77 \Omega) / (5 + j7.85 \Omega)}{-j6.77 \Omega} = -6.66 - j1.05 \text{ A}$$

$$\mathbf{I}_4 = (-2.99 + j2.20 \text{ A}) \frac{(-j6.77 \Omega) / (5 + j7.85 \Omega)}{5 + j7.85 \Omega} = 3.68 + j3.25 \text{ A}$$

Convert both of these values into time-varying signals.

$$i_3(t) = 6.75 \text{ A} \cos(2\pi 50t - 171.00\pi/180)$$

$$i_4(t) = 4.91 \text{ A} \cos(2\pi 50t + 41.48\pi/180)$$

9.1.3 KCL and KVL Analysis

Kirchhoff's current and voltage laws still hold in a phasor transformed circuit. KCL states that the sum of all branch currents entering a node is equal to zero. KVL states that the sum of all voltage drops in a loop

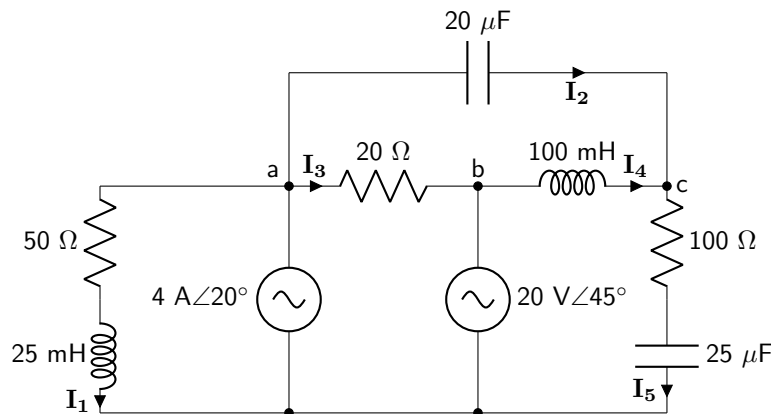
is equal to zero.

Together, KCL and KVL can be used to solve for every branch current in a circuit. Those branch currents can be multiplied by impedance to find voltage drops over every circuit element. By analyzing nodes and loops in a circuit, a number of equations (using KCL, KVL, and Ohm's law) can be found. If there are n unknown branch currents in the circuit, n linearly independent equations will be needed. A matrix can then be used to solve for the unknowns.

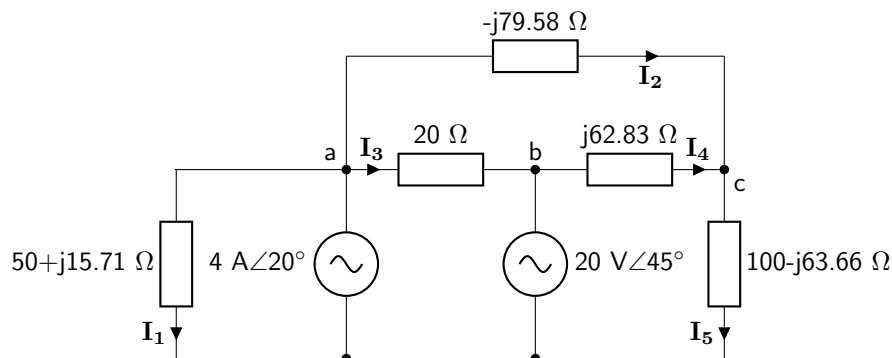
The equations that are derived to determine the matrix defining the circuit will now contain complex coefficients. Not all graphing calculators are capable of solving a matrix with complex coefficients. The TI-89 is capable of solving complex matrices. Otherwise, programs such as Matlab or websites such as [Wolfram Alpha](#) are capable of doing such calculations.

Example: Complex KCL / KVL analysis

Calculate all branch currents in the following circuit. All sources have a frequency of 100 Hz.



Phasor transform all circuit elements and re-draw.



The current source has a Cartesian form of $3.76 + j1.37$ A. The voltage source has a Cartesian form of $14.14 + j14.14$ V.

The circuit has two perfect nodes: node a and node c. These nodes will be solved using KCL. The

circuit will be analyzed using units of A, V, and Ω , but units will be removed from all variables to make the equations neater.

$$\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = (3.76 + j1.37)$$

$$\mathbf{I}_2 + \mathbf{I}_4 - \mathbf{I}_5 = 0$$

There are three perfect loops in the circuit. One contains \mathbf{I}_1 , \mathbf{I}_3 , and the voltage source. The second one contains the voltage source, \mathbf{I}_4 , and \mathbf{I}_5 . The third one contains \mathbf{I}_2 , \mathbf{I}_3 , and \mathbf{I}_4 . Apply KVL around each loop. Use the direction of the current to determine if the voltage is a rise or a drop. Voltage rises will be negative, and voltage drops will be positive.

$$-(50 + j15.71)\mathbf{I}_1 + (20)\mathbf{I}_3 = -(14.14 + j14.14)$$

$$(j62.83)\mathbf{I}_4 + (100 - j63.66)\mathbf{I}_5 = (14.14 + j14.14)$$

$$(-j79.58)\mathbf{I}_2 - (j62.83)\mathbf{I}_4 - (20)\mathbf{I}_3 = 0$$

Now that five linearly independent equations have been found, they can be put into the form of $\alpha\mathbf{I}_1 + \beta\mathbf{I}_2 + \gamma\mathbf{I}_3 + \delta\mathbf{I}_4 + \epsilon\mathbf{I}_5 = \mathbf{C}$ and placed into a matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & (3.76 + j1.37) \\ 0 & 1 & 0 & 1 & -1 & 0 \\ -(50 + j15.71) & 0 & 20 & 0 & 0 & -(14.14 + j14.14) \\ 0 & 0 & 0 & (j62.83) & (100 - j63.66) & (14.14 + j14.14) \\ 0 & (-j79.58) & -20 & -(j62.83) & 0 & 0 \end{bmatrix}$$

Solve for each current by finding the reduced row echelon form of the matrix.

$$\mathbf{I}_1 = 1.15 + j0.13 \text{ A}$$

$$\mathbf{I}_2 = 0.53 + j0.71 \text{ A}$$

$$\mathbf{I}_3 = 2.08 + j0.53 \text{ A}$$

$$\mathbf{I}_4 = -0.84 - j0.24 \text{ A}$$

$$\mathbf{I}_5 = -0.31 + j0.47 \text{ A}$$

Convert each branch current to a time-dependent (cosine) function.

$$i_1(t) = 1.16 \text{ A } \cos(2\pi 100t + 6.48\pi/180)$$

$$i_2(t) = 0.89 \text{ A } \cos(2\pi 100t + 53.31\pi/180)$$

$$i_3(t) = 2.14 \text{ A } \cos(2\pi 100t + 14.24\pi/180)$$

$$i_4(t) = 0.87 \text{ A } \cos(2\pi 100t - 164.09\pi/180)$$

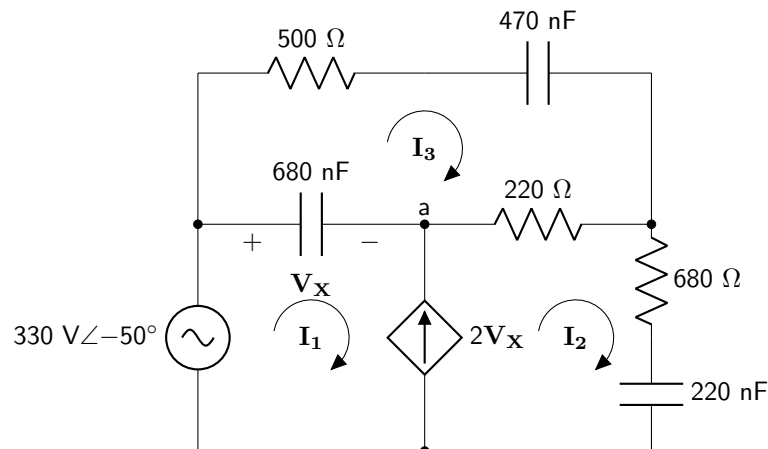
$$i_5(t) = 0.56 \text{ A } \cos(2\pi 100t + 123.23\pi/180)$$

9.1.4 Mesh Analysis

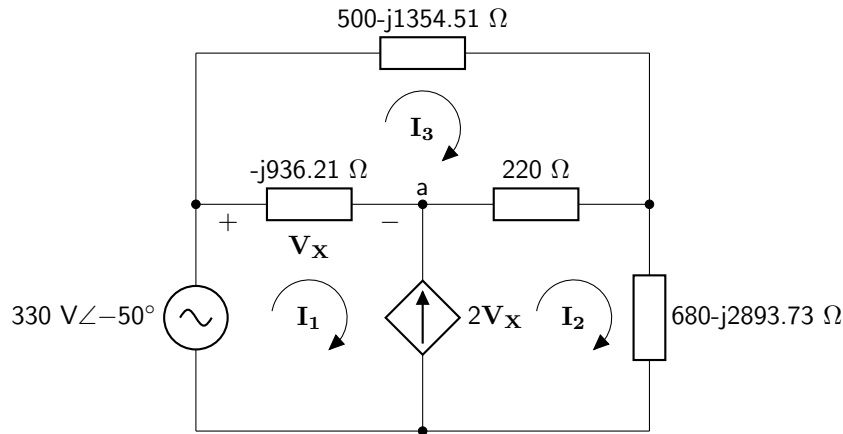
Mesh analysis can also be used in phasor transformed circuits. KCL and KVL are used to solve for mesh currents. The mesh analysis method steps, outlined in section 2.8.2 of this book, are exactly the same in a phasor transformed circuit.

Example: Complex mesh analysis

Calculate all mesh currents as well as \mathbf{V}_x in the following circuit. All sources have a frequency of 250 Hz. The proportionality constant of the VCCS has units of A/V.



Phasor transform all circuit elements and re-draw.



The voltage source has a Cartesian form of $212.12 - j252.79$ V.

There is only one regular mesh in this circuit: the one containing mesh current \mathbf{I}_3 . The circuit will be analyzed using units of A, V, and Ω , but units will be removed from all variables to make the equations neater.

$$(500 - j1354.51)\mathbf{I}_3 + (220)(\mathbf{I}_3 - \mathbf{I}_2) + (-j936.21)(\mathbf{I}_3 - \mathbf{I}_1) = 0$$

The two loops containing \mathbf{I}_1 and \mathbf{I}_2 are a supermesh. KCL will be applied at node a.

$$(-j936.21)(\mathbf{I}_1 - \mathbf{I}_3) + (220)(\mathbf{I}_2 - \mathbf{I}_3) + (680 - j2893.73)\mathbf{I}_2 = (212.12 - j252.79)$$

$$\mathbf{I}_1 + 2\mathbf{V}_X - \mathbf{I}_2 = 0$$

The presence of the controlling voltage (\mathbf{V}_X) provides an additional unknown. This equation will come from the relationship between the controlling voltage and mesh currents \mathbf{I}_1 and \mathbf{I}_3 .

$$\mathbf{V}_X = (-j936.21)(\mathbf{I}_1 - \mathbf{I}_3)$$

Put all four equations into the form of $\alpha\mathbf{I}_1 + \beta\mathbf{I}_2 + \gamma\mathbf{I}_3 + \delta\mathbf{V}_X = \mathbf{C}$ and place into a matrix.

$$\begin{bmatrix} (j936.21) & -220 & (720 - j2290.72) & 0 & 0 \\ (-j936.21) & (900 - j2893.73) & -(220 - j936.21) & 0 & (212.12 - j252.79) \\ 1 & -1 & 0 & 2 & 0 \\ (-j936.21) & 0 & (j936.21) & -1 & 0 \end{bmatrix}$$

Solve for each mesh current as well as the controlling voltage by finding the reduced row echelon form

of the matrix.

$$\mathbf{I}_1 = 0.001 + j0.02 \text{ A}$$

$$\mathbf{I}_2 = 0.10 + j0.04 \text{ A}$$

$$\mathbf{I}_3 = 0.001 + j0.02 \text{ A}$$

$$\mathbf{V}_X = 0.05 + j0.01 \text{ V}$$

Convert each variable to a time-dependent (cosine) function. (Note the units have been changed to mA and mV due to the small quantities.)

$$i_1(t) = 15.59 \text{ mA} \cos(2\pi 100t + 85.20\pi/180)$$

$$i_2(t) = 108.09 \text{ mA} \cos(2\pi 100t + 23.14\pi/180)$$

$$i_3(t) = 15.54 \text{ mA} \cos(2\pi 100t + 85.13\pi/180)$$

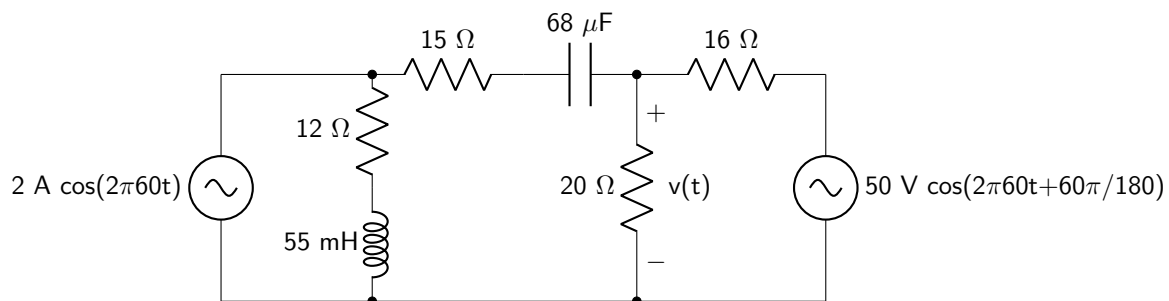
$$v_X(t) = 50.86 \text{ mV} \cos(2\pi 100t + 15.36\pi/180)$$

9.1.5 Superposition

Superposition applies to all linear circuits. If a linear circuit has n independent sources, then n subcircuits can be created, each with one of the independent sources activated and all others deactivated. Add the properties of each of the subcircuits together to find the total value of the property. To deactivate a voltage source, replace the voltage source with a short circuit (corresponding to 0 V). To deactivate a current source, replace the current source with an open circuit (corresponding to 0 A). Any dependent sources that exist in the circuit cannot be eliminated and must be present in all subcircuits.

Example: Complex superposition

Use superposition to calculate $v(t)$ in the circuit below.



Calculate the impedances of the inductor and the capacitor.

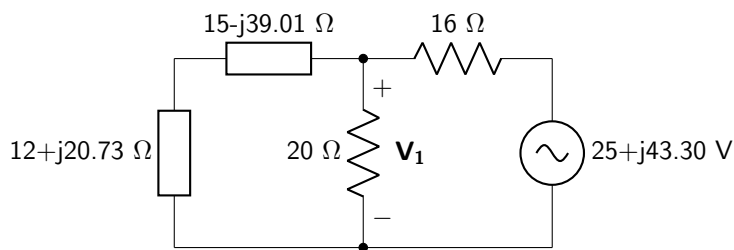
$$\mathbf{Z}_L = j2\pi 60(0.055) = j20.73 \Omega$$

$$\mathbf{Z}_C = -\frac{j}{2\pi 60(68 \times 10^{-6})} = -j39.01 \Omega$$

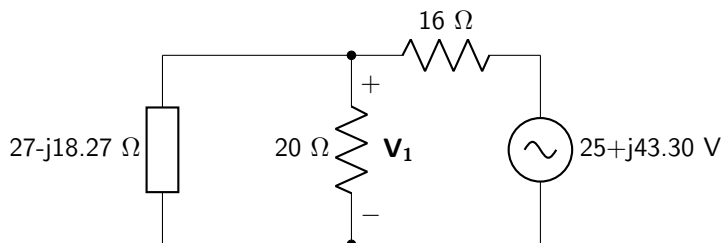
Both sources can be converted to phasor form. The current source is 2 A in phasor form because the angle is zero degrees.

$$\mathbf{V} = 50 \cos 60^\circ + j50 \sin 60^\circ \text{ V} = 25 + j43.30 \text{ V}$$

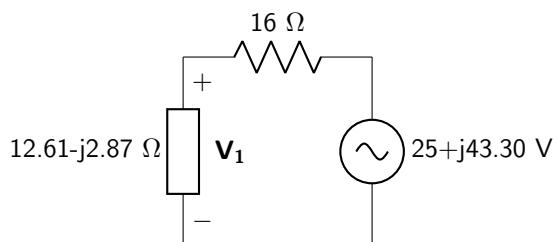
Deactivate the current source and calculate the contribution to the output voltage provided by the voltage source.



The two impedances are in series and can be combined together.



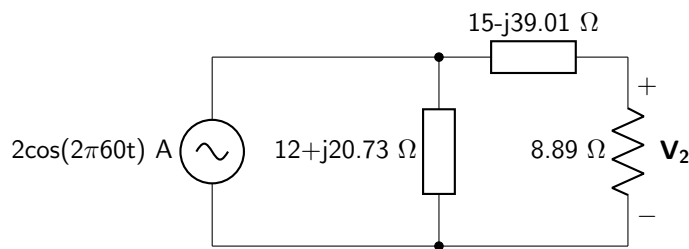
The 20 Ω resistor can be combined in parallel with the impedance.



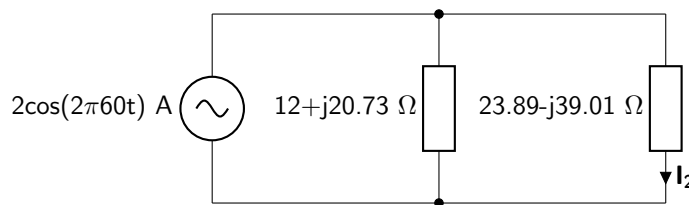
Use the voltage divider rule to calculate \mathbf{V}_1 .

$$\mathbf{V}_1 = (25 + j43.30 \text{ V}) \frac{12.61 - j2.87 \Omega}{16 \Omega + (12.61 - j2.87 \Omega)} = 13.57 + j17.93 \text{ V}$$

Deactivate the voltage source and calculate the contribution to the output voltage provided by the current source. The 16Ω and 20Ω resistors have been combined in parallel.



Combine the $15 - j39.01 \Omega$ impedance in series with the 8.89Ω resistor so that a current divider can be used.



Use the current divider rule to calculate \mathbf{I}_2 .

$$\mathbf{I}_2 = (2 \text{ A}) \frac{(12 + j20.73 \Omega) / (23.89 - j39.01 \Omega)}{23.89 - j39.01 \Omega} = 0.06 + j1.19 \text{ A}$$

Use Ohm's law to calculate \mathbf{V}_2 .

$$\mathbf{V}_2 = (8.89 \Omega)(0.06 + j1.19 \text{ A}) = 0.57 + j10.56 \text{ V}$$

Add together the two voltages to obtain the total voltage \mathbf{V} .

$$\mathbf{V} = (13.57 + j17.93 \text{ V}) + (0.57 + j10.56 \text{ V}) = 14.13 + j28.49 \text{ V}$$

Convert this value into a time-varying signal.

$$v(t) = 31.81 \text{ V} \cos(2\pi 60t + 63.62\pi/180)$$

9.1.6 Source Transformation

In a phasor transformed circuit, source transformation can be used to convert an AC current source into an AC voltage source and vice versa. The transformation process changes the source type and the impedance location, but results in an equivalent circuit to the original. The goal of source transformation is usually to reduce a circuit by changing the position of the source and an impedance; this can lead to further reductions using series and parallel combinations of impedances.

Two equivalent AC circuits, as shown in figure 9.1, will have identical voltage drops over the load (depicted as \mathbf{V}) as well as identical currents flowing into the load (depicted as \mathbf{I}).

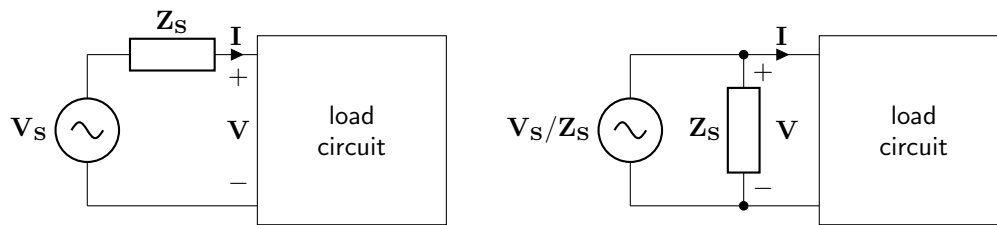


Figure 9.1: Both of these circuits are equivalent as they have identical voltage drops over the load (\mathbf{V}) and current flow into the load (\mathbf{I}).

In this manner, a voltage source in series with an impedance can be made equivalent to a current source in parallel with an impedance. The complex version of Ohm's law is used to determine the value of the voltage or current source. When transforming a voltage source to a current source, the current source value will be equal to $\mathbf{V}_S/\mathbf{Z}_S$. This is depicted schematically in figure 9.2.

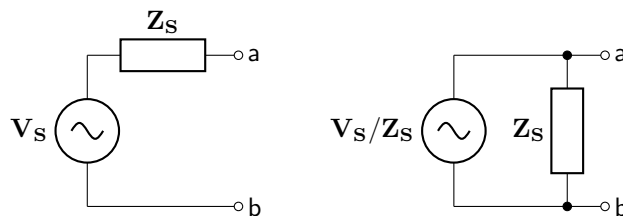


Figure 9.2: The transformation of a voltage source in series with an impedance (left) to a current source in parallel with an impedance (right).

When transforming a current source to a voltage source, the voltage source value will be equal to $\mathbf{I}_S\mathbf{Z}_S$ (using Ohm's law). This is depicted schematically in figure 9.3.

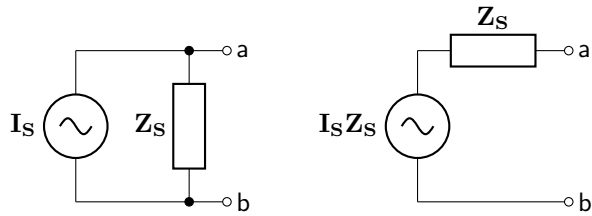
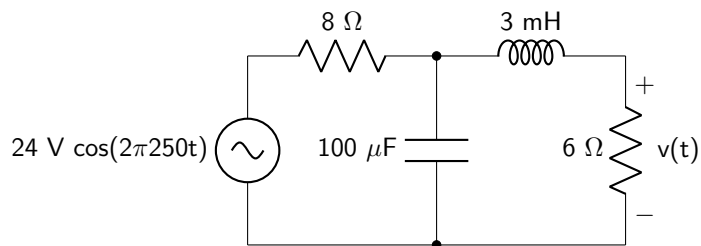


Figure 9.3: The transformation of a current source in parallel with an impedance (left) to a voltage source in series with an impedance (right).

Example: Complex source transformation

Use source transformation and a voltage divider to calculate $v(t)$.



Calculate the impedances of the inductor and the capacitor.

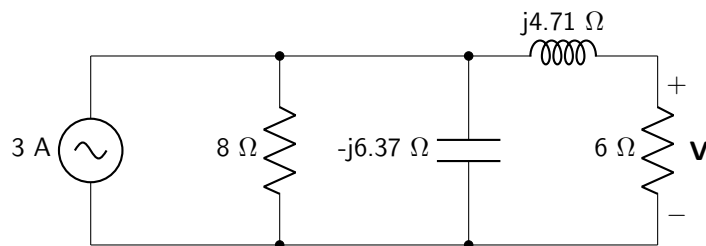
$$\mathbf{Z}_L = j2\pi 250(0.003) = j4.71 \Omega$$

$$\mathbf{Z}_C = -\frac{j}{2\pi 250(100 \times 10^{-6})} = -j6.37 \Omega$$

Convert the source to a phasor. Because the angle is zero degrees, the phasor form is 24 V.

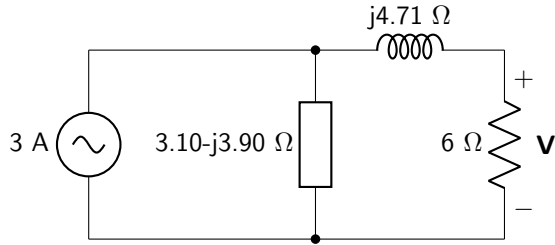
Source transform the voltage source to a current source. The value of the current source will be 3 A.

The circuit can be re-drawn.



The 8Ω resistor and the capacitor can be combined in parallel.

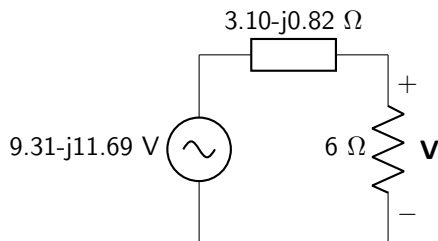
$$\mathbf{Z} = \frac{(8 \Omega)(-j6.37 \Omega)}{(8 \Omega) + (-j6.37 \Omega)} = 3.10 - j3.90 \Omega$$



Perform another source transformation. Use Ohm's law to calculate the value of the voltage source.

$$\mathbf{V} = (3 \text{ A})(3.10 - j3.90 \Omega) = 9.31 - j11.69 \text{ V}$$

The circuit can be re-drawn. The impedance of the inductor has been added to the impedance of the source resistor.



Use a voltage divider to calculate \mathbf{V} .

$$\mathbf{V} = (9.31 - j11.69 \text{ V}) \frac{6 \Omega}{6 \Omega + (3.10 - j0.82 \Omega)} = 5.40 - j8.19 \text{ V}$$

Convert this value into a time-varying signal.

$$v(t) = 9.81 \text{ V} \cos(2\pi 250t - 56.60\pi/180)$$

9.1.7 Thévenin and Norton's Theorems

The phasor transformed version of Thévenin's theorem states that any linear circuit (regardless of the complexity) can be represented by an equivalent circuit that contains a voltage source in series with an impedance. This is depicted in figure 9.4.

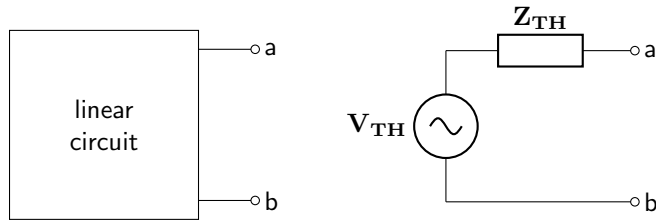


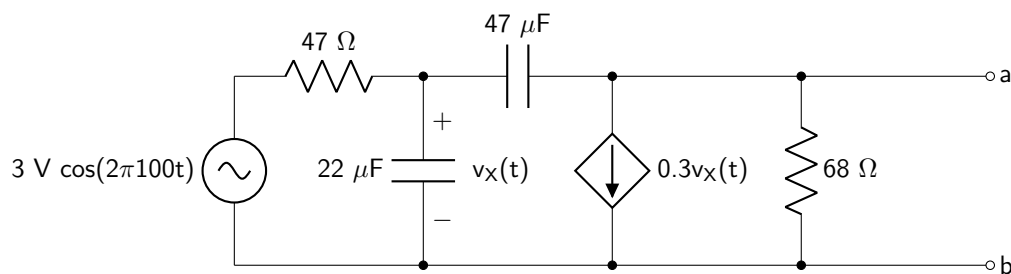
Figure 9.4: Thévenin's theorem states that these two circuits are equivalent, given the correct value of V_{TH} and Z_{TH} .

The Thévenin equivalent voltage (V_{TH}) is equal to the open-circuit voltage between the output terminals. The Thévenin equivalent impedance (Z_{TH}) is solved using the procedure below.

- If there are no dependent sources in the circuit, then Z_{TH} is equal to the equivalent impedance seen between the output terminals, with all of the independent sources deactivated.
- If there are dependent sources in the circuit, calculate the short-circuit current (I_{SC}) between the output terminals. The Thévenin equivalent impedance is equal to V_{TH}/I_{SC} .

Example: Complex Thévenin equivalent circuit

Find a Thévenin equivalent circuit.



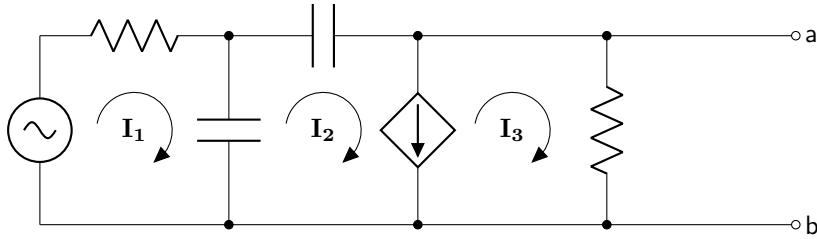
Calculate the impedance of both capacitors.

$$Z_{22\mu\text{F}} = -\frac{j}{2\pi 250(22 \times 10^{-6})} = -j72.34 \Omega$$

$$Z_{47\mu\text{F}} = -\frac{j}{2\pi 250(47 \times 10^{-6})} = -j33.86 \Omega$$

Convert the source to a phasor. Because the angle is zero degrees, the phasor form is 3 V.

Mesh analysis will be used to solve this circuit. The mesh currents are depicted below on a simplified circuit diagram.



Perform KVL around mesh 1. (At this point, units will be removed from most equations. They will all be in V, Ω , and A, unless otherwise specified.)

$$3 = 47\mathbf{I}_1 + (-j72.34)(\mathbf{I}_1 - \mathbf{I}_2)$$

Perform KVL around the supermesh.

$$0 = (-j72.34)(\mathbf{I}_2 - \mathbf{I}_1) + (-j33.86)\mathbf{I}_2 + 68\mathbf{I}_3$$

KCL is performed at the intersection of meshes two and three.

$$0 = \mathbf{I}_2 - 0.3\mathbf{V}_X - \mathbf{I}_3$$

An equation can be created to define the dependent source in terms of mesh currents.

$$\mathbf{V}_X = (-j72.34)(\mathbf{I}_1 - \mathbf{I}_2)$$

Create a matrix in terms of $\alpha\mathbf{I}_1 + \beta\mathbf{I}_2 + \gamma\mathbf{I}_3 + \delta\mathbf{V}_X = \epsilon$.

$$\begin{bmatrix} 47 - j72.34 & j72.34 & 0 & 0 & 3 \\ j72.34 & -j106.21 & 68 & 0 & 0 \\ 0 & 1 & -1 & -0.3 & 0 \\ j72.34 & -j72.34 & 0 & 1 & 0 \end{bmatrix}$$

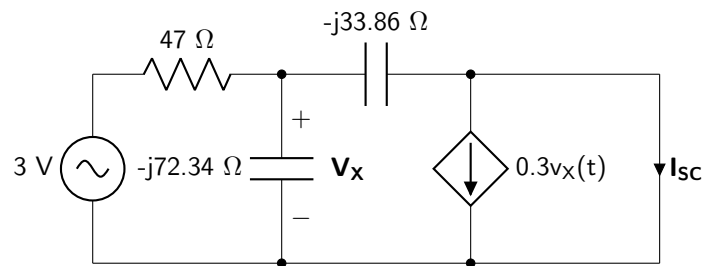
Reduce the matrix to reduced row echelon form. The quantity of interest is \mathbf{I}_3 . When multiplied by 68Ω it yields \mathbf{V}_{TH} .

$$\mathbf{V}_{TH} = ((3 + j27.78) \times 10^{-3} \text{ A}) (68 \Omega) = 0.20 + j1.89 \text{ V}$$

This quantity can be converted to time-varying form.

$$v_{th}(t) = 1.9 \text{ V} \cos(2\pi 100t + 83.84\pi/180)$$

Because there is a dependent source, the short-circuit current must be calculated to determine the value of the Thévenin equivalent impedance. The $68\ \Omega$ resistor is shorted and can be removed from the circuit.



Mesh analysis will be used to solve this circuit. The meshes will be defined the same as above. Perform KVL around mesh 1.

$$3 = 47\mathbf{I}_1 + (-j72.34)(\mathbf{I}_1 - \mathbf{I}_2)$$

Perform KVL around the supermesh.

$$0 = (-j72.34)(\mathbf{I}_2 - \mathbf{I}_1) + (-j33.86)\mathbf{I}_2$$

KCL is performed at the intersection of meshes two and three.

$$0 = \mathbf{I}_2 - 0.3\mathbf{V}_X - \mathbf{I}_3$$

An equation can be created to define the dependent source in terms of mesh currents.

$$\mathbf{V}_X = (-j72.34)(\mathbf{I}_1 - \mathbf{I}_2)$$

Create a matrix in terms of $\alpha\mathbf{I}_1 + \beta\mathbf{I}_2 + \gamma\mathbf{I}_3 + \delta\mathbf{V}_X = \epsilon$.

$$\begin{bmatrix} 47 - j72.34 & j72.34 & 0 & 0 & 3 \\ j72.34 & -j106.21 & 0 & 0 & 0 \\ 0 & 1 & -1 & -0.3 & 0 \\ j72.34 & -j72.34 & 0 & 1 & 0 \end{bmatrix}$$

Reduce the matrix to reduced row echelon form. The quantity of interest is \mathbf{I}_3 , which is equal to \mathbf{I}_{SC} .

$$\mathbf{I}_{SC} = 139.65 + j373.15\ \text{mA}$$

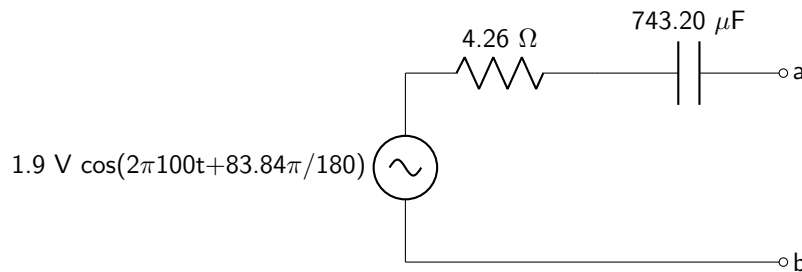
Use Ohm's law to calculate the Thévenin equivalent impedance.

$$\mathbf{Z}_{\text{TH}} = \frac{0.20 + j1.89 \text{ V}}{0.14 + j0.37 \text{ A}} = 4.26 - j2.14 \ \Omega$$

This is equal to a resistor of $4.26 \ \Omega$ in series with a capacitor. The capacitor value can be calculated.

$$C = \frac{1}{2\pi 100(2.14)} = 743.20 \ \mu\text{F}$$

The Thévenin equivalent circuit is shown below.



Norton's theorem states that any linear circuit, regardless of the level of complexity, can be represented by an equivalent circuit that contains a current source in parallel with an impedance. This is depicted in figure 9.5.

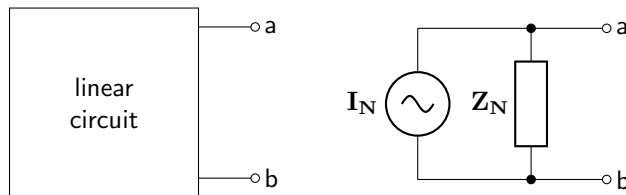


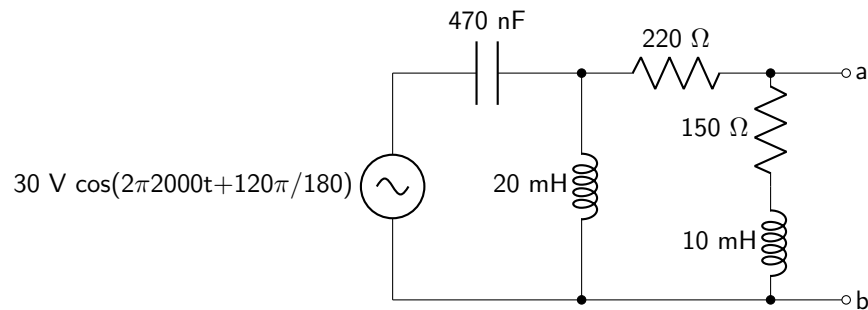
Figure 9.5: Norton's theorem states that these two circuits are equivalent, given the correct value of \mathbf{I}_{N} and \mathbf{Z}_{N} .

Norton's theorem is simply a source transformed version of a Thévenin equivalent circuit. However, it is possible to derive a Norton equivalent circuit without first deriving a Thévenin equivalent circuit. The Norton equivalent current (\mathbf{I}_{N}) is equal to the short-circuit current between the output terminals. The procedure to calculate the Norton equivalent impedance is given below.

- If there are no dependent sources in the circuit, then \mathbf{Z}_{N} is equal to the equivalent impedance seen between the output terminals, with all of the independent sources deactivated.
- If there are dependent sources in the circuit, calculate the open-circuit voltage (\mathbf{V}_{OC}) between the output terminals. The Norton equivalent impedance is equal to $\mathbf{V}_{\text{OC}}/\mathbf{I}_{\text{N}}$.

Example: Complex Norton equivalent circuit

Find a Norton equivalent circuit.



Calculate the impedance of both inductors and the capacitor, and convert the voltage source to phasor form.

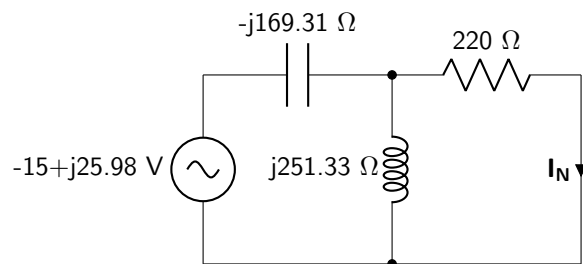
$$\mathbf{Z}_C = -\frac{j}{2\pi 2000(470 \times 10^{-9})} = -j169.31 \Omega$$

$$\mathbf{Z}_{20\text{mH}} = j2\pi 2000(0.02) = j251.33 \Omega$$

$$\mathbf{Z}_{10\text{mH}} = j2\pi 2000(0.01) = j125.66 \Omega$$

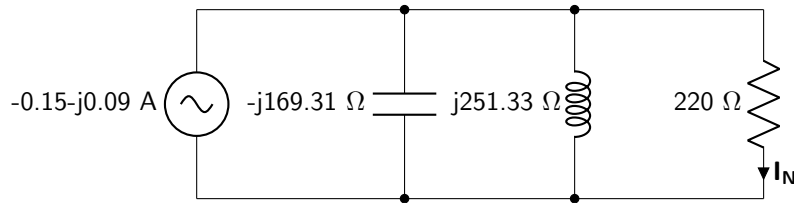
$$\mathbf{V} = 30 \cos 120^\circ + j30 \sin 120^\circ = -15 + j25.98 \text{ V}$$

Calculate the Norton equivalent current. Because the 150 Ω resistor and 10 mH inductor are shorted, they are effectively removed from the circuit.



Perform a source transformation. The value of the current source is calculated below.

$$\mathbf{I} = \frac{-15 + j25.98 \text{ V}}{-j169.31 \Omega} = -0.15 - j0.09 \text{ A}$$



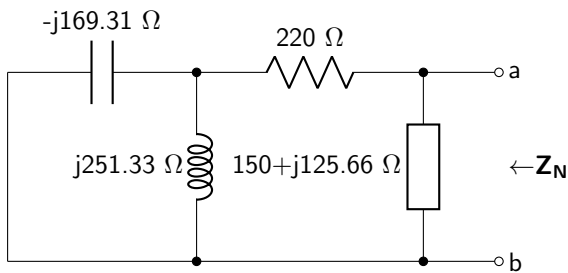
Use a current divider to calculate I_N .

$$I_N = (-0.15 - j0.09 \text{ A}) \frac{(-j169.31 \Omega) // (j251.33 \Omega) // (220 \Omega)}{220 \Omega} = -0.16 - j0.02 \text{ A}$$

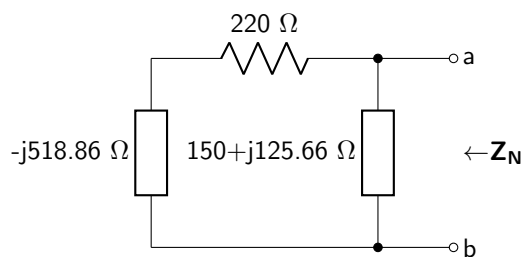
This quantity can be converted to time-varying form.

$$i_n(t) = 163.13 \text{ mA} \cos(2\pi 2000t - 172.98\pi/180)$$

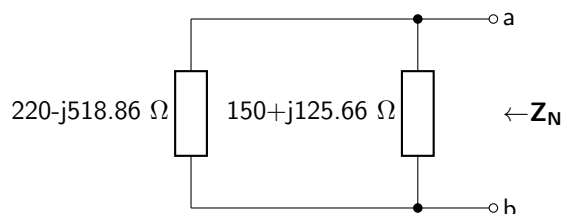
There are no dependent sources, so the Norton equivalent impedance can be calculated directly. The voltage source will be deactivated.



Combine the capacitor and inductor impedances in parallel.



Add that impedance to 220 Ω.



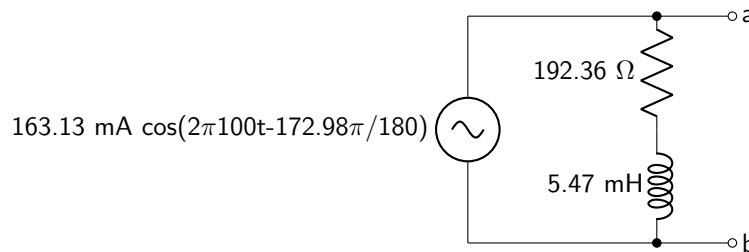
Combine both impedances in parallel.

$$\mathbf{Z}_N = 192.36 + j68.76 \, \Omega$$

This is equal to a resistor of $192.36 \, \Omega$ in series with an inductor. The inductor value can be calculated.

$$L = \frac{68.76}{2\pi 2000} = 5.47 \, \text{mH}$$

The Norton equivalent circuit is shown below.



9.2 Passive Filters

A filter is a circuit that selectively blocks certain frequencies (or a range of frequencies) from passing through to the output. Filters can be passive (made of RLC components only) or active (made with RLC components and an op-amp). Active filters will be discussed in section 9.3.

Filters are defined by their frequency response. The four types of frequency responses are low-pass, high-pass, band-pass, and band-stop. Each of these four passive filter types will be explained in sections 9.2.1–9.2.4. In addition, filters are defined by the circuit order. First order filters consist of first order circuits (only one energy storage device is included in the design). Second order filters consist of second order circuits (two energy storage devices are included in the design).

9.2.1 Low-Pass Filters (LPF)

A low-pass filter blocks high frequency signals (above a cutoff frequency) from passing through to the output. (Only low frequencies can pass through, hence the name “low-pass filter.”) Low-pass filters are useful for removing high frequency noise (for example) from a circuit’s output.

As the input voltage frequency changes, the magnitude of the output voltage will change. This can be characterized by calculating v_{out}/v_{in} at varying frequencies and looking at the response. An ideal low-pass filter has what’s called a “brick wall” response: all frequencies above the cutoff frequency are completely eliminated from the output, and all frequencies below the cutoff frequency are able to pass through with no change in magnitude. This ideal brick wall output response, as well as a real low-pass filter response, is shown in figure 9.6 (note the use of the log/log scale).

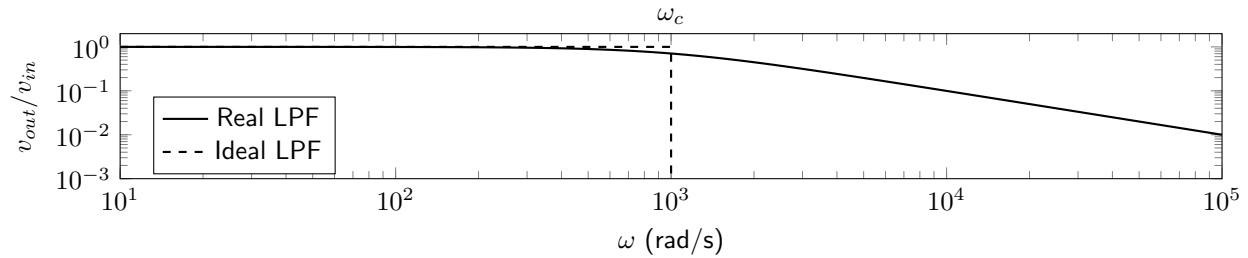


Figure 9.6: Comparison between an ideal (“brick wall”) low-pass filter and a real low-pass filter.

There are many different ways to implement a low-pass filter. One example of a first order passive low-pass filter is shown in figure 9.7. The capacitor, having a low impedance at high frequencies, leads to a decreasing voltage drop as frequency increases.

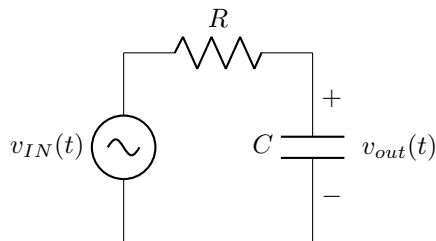


Figure 9.7: This resistor-capacitor circuit acts as a low-pass filter. As frequency increases, the voltage drop over the load will decrease.

It is also possible to use a resistor-inductor combination to create a first order low-pass filter. This circuit, shown in figure 9.8, acts as a low-pass filter because the inductor acts as an open at high frequencies, blocking current flow through the output resistor.

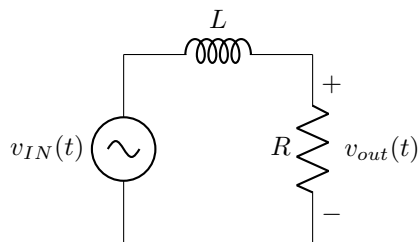


Figure 9.8: This resistor-inductor circuit acts as a low-pass filter. As frequency increases, current flow through the load will decrease.

Both filters in figures 9.7 and 9.8 are first order circuits. The cutoff frequency of a first order filter is given by equation 9.3.

$$\omega_c = a_0 \quad (9.3)$$

Second order passive low-pass filters can be created by cascading two first order filters together, as shown

in figure 9.9.

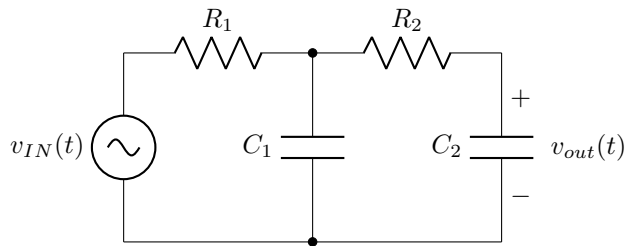


Figure 9.9: This circuit acts as a second order low-pass filter by cascading together two resistor-capacitor filters.

The cutoff frequency of a second order filter is given by equation 9.4. This is equal to the resonant frequency of the second order circuit.

$$\omega_c = \sqrt{a_0} \quad (9.4)$$

Other second order filter architectures include cascading RL filters, or by using a series RLC circuit. Because of parasitic resistance effects in inductors, it is not always practical to use inductor-based filters.

A second order filter will cause a much faster decay of the output signal compared to a first order filter. A comparison of waveforms for a first and second order RC low-pass filter (with identical cutoff frequencies) is shown in figure 9.10.

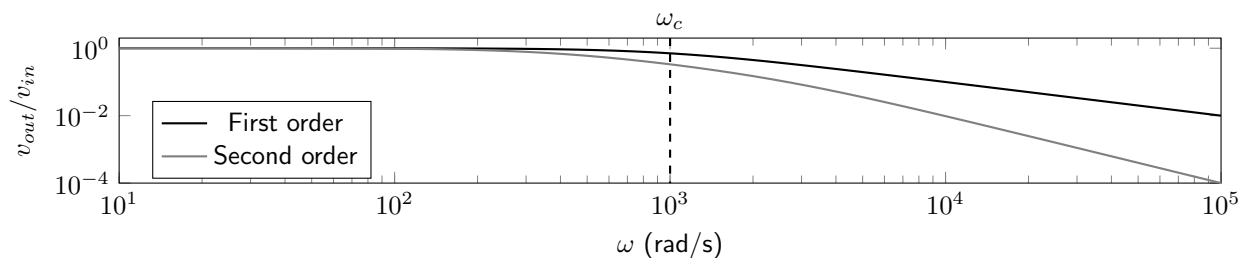
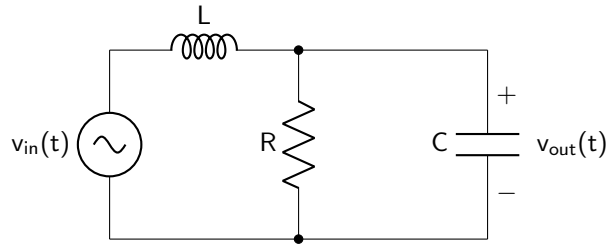


Figure 9.10: Comparison between first order RC and second order RC low-pass filter.

There is theoretically no limit to how many stages can be cascaded together to create higher-order filters. This textbook will only consider first and second order circuits.

Example: Low-pass filter analysis in symbolic form

Analyze the following low-pass filter. Determine the cutoff frequency



The second order equation will provide information about the properties of the circuit. It can be found by performing KCL at the node between the inductor, resistor, and capacitor.

$$\frac{1}{LC}v_{in}(t) = \frac{d^2v_{out}(t)}{dt^2} + \frac{1}{RC}\frac{dv_{out}(t)}{dt} + \frac{1}{LC}v_{out}(t)$$

Using equation 9.4, it is now possible to calculate the cutoff frequency of the circuit.

$$\omega_c = \frac{1}{\sqrt{LC}}$$

9.2.2 High-Pass Filters (HPF)

A high-pass filter blocks low frequency signals (below a cutoff frequency) from passing through to the output. (Only high frequencies can pass through, hence the name “high-pass filter.”)

An ideal high-pass filter has a brick wall response: all frequencies below the cutoff frequency are completely eliminated from the output, and all frequencies above the cutoff frequency are able to pass through with no change in magnitude. This ideal brick wall output response, as well as a real high-pass filter response, is shown in figure 9.11.

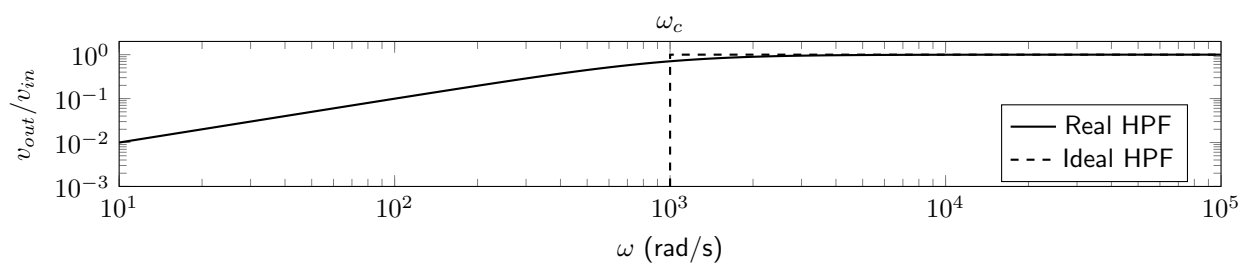


Figure 9.11: Comparison between an ideal (“brick wall”) high-pass filter and a real high-pass filter.

There are many different ways to implement a high-pass filter. One example of a first order passive high-pass filter is shown in figure 9.12. The capacitor, having a high impedance at low frequencies, blocks current from passing through the load at low frequencies.

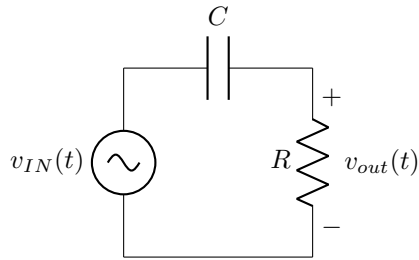


Figure 9.12: This resistor-capacitor circuit acts as a high-pass filter. At low frequencies, current cannot pass through the load resistor.

It is also possible to use a resistor-inductor combination to create a first order high-pass filter. This circuit, shown in figure 9.13, acts as a high-pass filter because the inductor acts as a short at low frequencies, creating leading to a small voltage drop over the load.

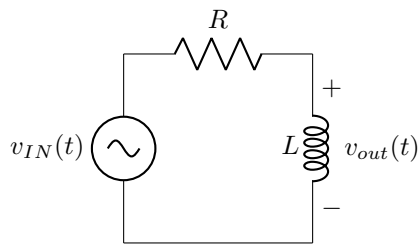


Figure 9.13: This resistor-inductor circuit acts as a high-pass filter. At low frequencies, there will be minimal voltage dropped over the load.

The cutoff frequency for a high-pass first order filter is the same as that of a first order low-pass filter. The relationship is given in equation 9.3.

It is possible to create higher order high-pass filters. Second order filters can be created by cascading two first order filters together, as shown in figure 9.14.

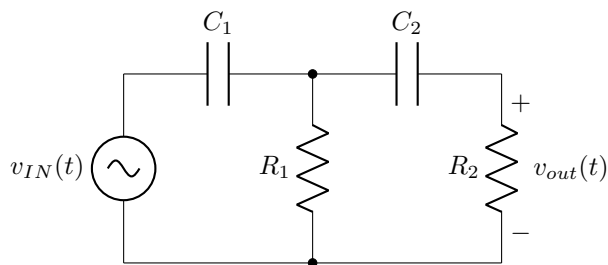


Figure 9.14: This circuit acts as a second order high-pass filter by cascading together two resistor-capacitor filters.

The cutoff frequency of a second order filter is given by equation 9.4, and is equal to the resonant frequency of the second order circuit. There are other possible architectures for creating passive second order high-pass filters.

A second order filter will cause a much faster decay of the output signal compared to a first order filter.

A comparison of waveforms for a first and second order RC high-pass filter (with identical cutoff frequencies) is shown in figure 9.15.

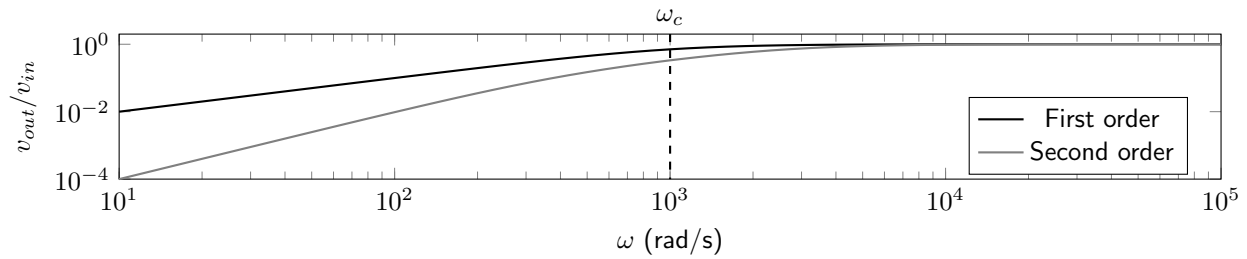
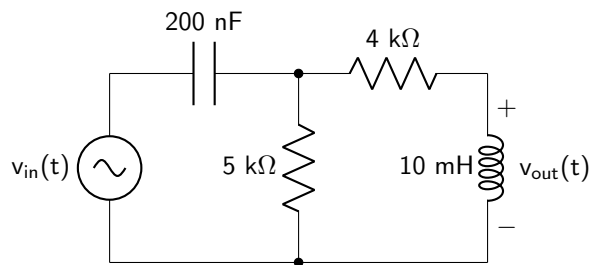


Figure 9.15: Comparison between first order RC and second order RC high-pass filter.

There is theoretically no limit to how many stages can be cascaded together to create higher-order filters. This textbook will only consider first and second order circuits.

Example: High-pass filter analysis

Analyze the following high-pass filter. Calculate the cutoff frequency.



The second order equation will provide information about the properties of the circuit. It can be derived using any circuit analysis technique.

$$\frac{d^2 v_{in}(t)}{dt} = \frac{d^2 v_{out}(t)}{dt^2} + (401000) \frac{dv_{out}(t)}{dt} + (9 \times 10^8) v_{out}(t)$$

Using equation 9.4, it is now possible to calculate the cutoff frequency of the circuit.

$$\omega_c = 30000 \text{ rad/s}$$

9.2.3 Band-Pass Filters (BPF)

A band-pass filter only allows a range of frequencies to pass through to the output. Anything below a certain frequency, or above a certain frequency, is blocked from reaching the output.

An ideal band-pass filter has a brick wall response: all frequencies within the so-called passband (centered around the center frequency ω_0) are able to pass through with no change in magnitude. All frequencies

outside of the passband are eliminated. This ideal brick wall output response, as well as a real band-pass filter response, is shown in figure 9.16.

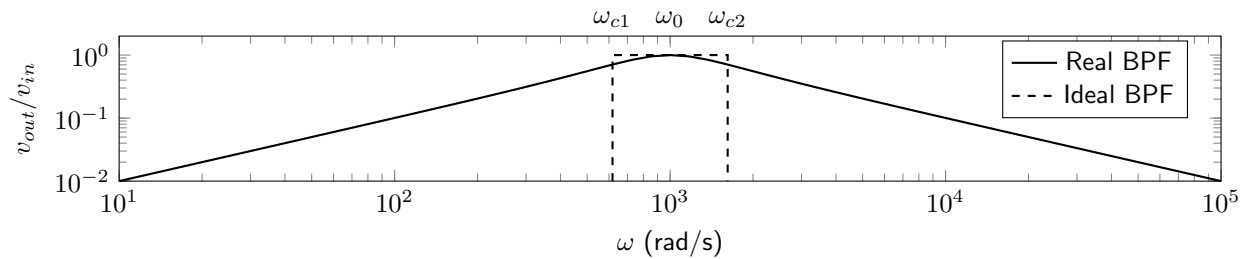


Figure 9.16: Comparison between an ideal (“brick wall”) band-pass filter and a real band-pass filter.

At least two energy-storage elements are required to create a band-pass filter; all band-pass filters must be at least second order circuits. One possible band-pass filter architecture is shown in figure 9.17. At very low frequencies, the inductor acts as a short and the capacitor acts as an open; there will be minimal voltage dropped over the load. At very high frequencies, the inductor acts as an open and the capacitor acts as a short; there will again be minimal voltage dropped over the load.

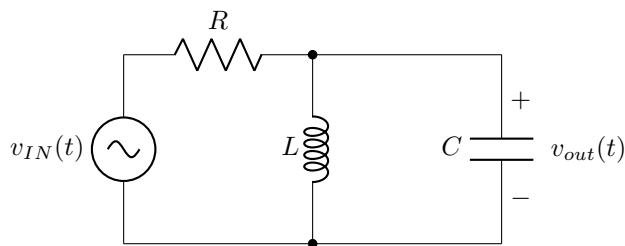


Figure 9.17: This RLC circuit acts as a band-pass filter. At both very low and very high frequencies, there will be minimal voltage dropped over the load.

The center frequency (ω_0) of a second order filter is equal to its resonant frequency, described in equation 7.3. The cutoff frequencies of a band-pass filter describe the frequency at which the voltage is reduced to a value of $1/\sqrt{2}$. There are two cutoff frequencies: ω_{c1} describes the low-frequency bound of the passband; ω_{c2} describes the high-frequency bound of the passband. Both cutoff frequencies are described in equations 9.5 and 9.6.

$$\omega_{c1} = -\alpha + \sqrt{\alpha^2 + \omega_0^2} \quad (9.5)$$

$$\omega_{c2} = \alpha + \sqrt{\alpha^2 + \omega_0^2} \quad (9.6)$$

The *bandwidth* of the circuit describes the width of the passband, and is described in equation 9.7. The wider the passband (and hence the larger the bandwidth), the more frequencies that are allowed to pass

through to the output.

$$\beta = \omega_{c2} - \omega_{c1} \quad (9.7)$$

Band-pass filters can also be characterized by their *quality factor*. A filter with a high quality factor will have a very narrow passband (low bandwidth), and vice versa. A graph showing the output response of band-pass filters with different quality factors is shown in figure 9.18.

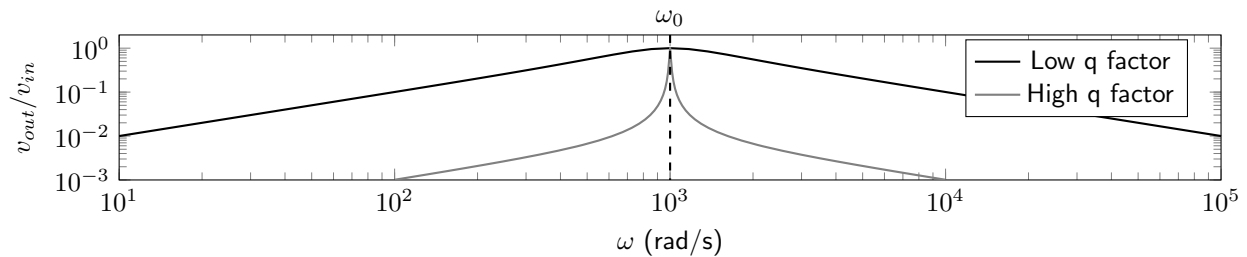


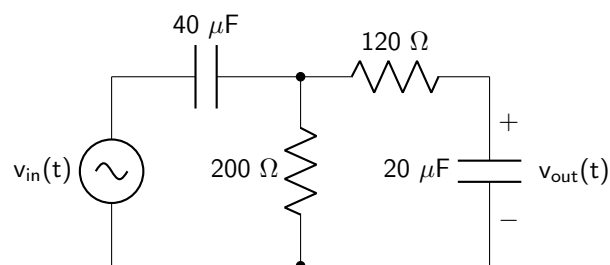
Figure 9.18: Comparison between band-pass filters with low and high quality factors.

The quality factor of a second order band-pass filter can be calculated based on its characteristic equation. This relationship is given in equation 9.8.

$$Q = \frac{\sqrt{a_0}}{a_1} \quad (9.8)$$

Example: Band-pass filter analysis

Analyze the following band-pass filter. Calculate the center frequency, both cutoff frequencies, the bandwidth, and the quality factor.



The second order equation will provide information about the properties of the circuit. It can be derived using any circuit analysis technique.

$$(416.67) \frac{dv_{in}(t)}{dt} = \frac{d^2v_{out}(t)}{dt^2} + (750) \frac{dv_{out}(t)}{dt} + (52083.33)v_{out}(t)$$

Using equation 9.4, it is now possible to calculate the center frequency of the circuit.

$$\omega_0 = 228.22 \text{ rad/s}$$

The cutoff frequencies can be calculated using equations 9.5 and 9.6.

$$\omega_{c1} = 63.99 \text{ rad/s}$$

$$\omega_{c2} = 813.99 \text{ rad/s}$$

Calculate the bandwidth using equation 9.7.

$$\beta = 750 \text{ rad/s}$$

Use equation 9.8 to calculate the quality factor.

$$Q = 0.304$$

9.2.4 Band-Stop Filters (BSF)

A band-stop filter rejects a range of frequencies centered around the center frequency. Anything below a certain frequency, or above a certain frequency, is passed through to the output.

An ideal band-stop filter has a brick wall response: all frequencies within the so-called stopband (centered around the center frequency) are rejected. All frequencies outside of the stopband are passed through. This ideal brick wall output response, as well as a real band-stop filter response, is shown in figure 9.19.

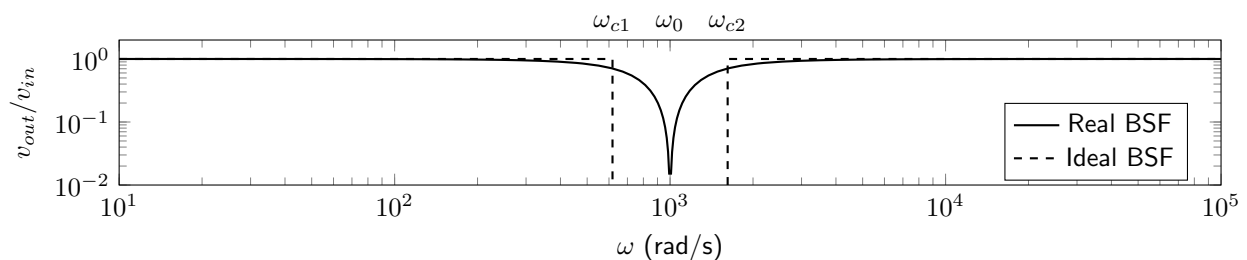


Figure 9.19: Comparison between an ideal (“brick wall”) band-stop filter and a real band-stop filter.

At least two energy-storage elements are required to create a band-stop filter; all band-stop filters must be at least second order circuits. One possible band-stop filter architecture is shown in figure 9.20. At very low frequencies, the inductor acts as a short and the capacitor acts as an open; there will be maximum voltage dropped over the load. At very high frequencies, the inductor acts as an open and the capacitor acts as a short; there will again be maximum voltage dropped over the load.

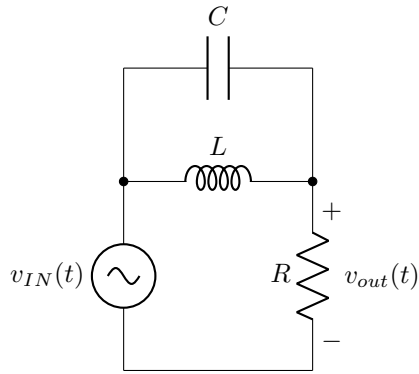


Figure 9.20: This RLC circuit acts as a band-stop filter. At both very low and very high frequencies, there will be maximum voltage dropped over the load.

The center frequency of a second order filter is given by its resonant frequency described in equation 7.3. Similarly, the cutoff frequencies can be described by equations 9.5 and 9.6. The bandwidth is given in equation 9.7.

Band-stop filters can also be characterized by their quality factor. A filter with a high quality factor will have a very narrow stopband, and vice versa. A graph showing the output response of band-stop filters with different quality factors is shown in figure 9.21.

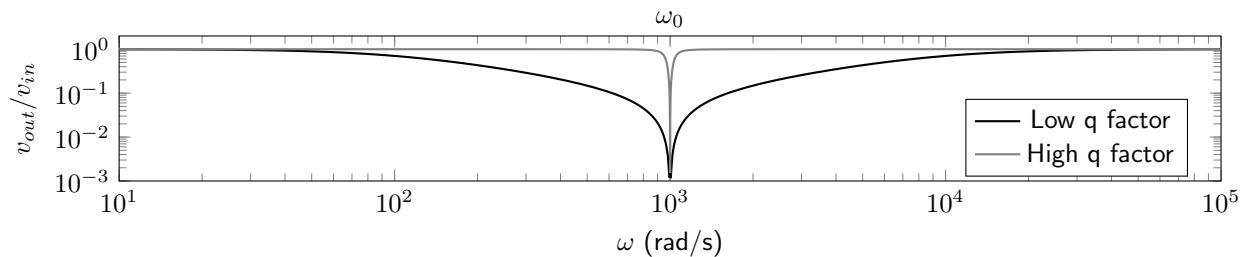
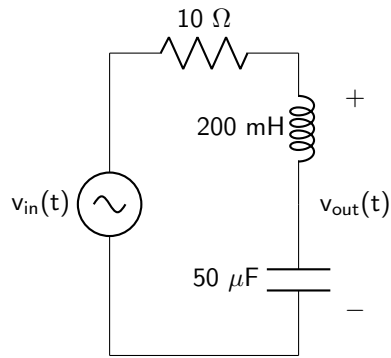


Figure 9.21: Comparison between band-stop filters with low and high quality factors.

The quality factor of a second order band-pass filter can be calculated based on its characteristic equation. This relationship is given in equation 9.8. It is not different from the equation used to find the quality factor in a band-pass filter.

Example: Band-stop filter analysis

Analyze the following band-stop filter. Calculate the center frequency, both cutoff frequencies, the bandwidth, and the quality factor.



The second order equation will provide information about the properties of the circuit. It can be derived using any circuit analysis technique.

$$\frac{d^2 v_{in}(t)}{dt^2} + 100000 v_{in}(t) = \frac{d^2 v_{out}(t)}{dt^2} + (50) \frac{dv_{out}(t)}{dt} + (100000) v_{out}(t)$$

Using equation 9.4, it is now possible to calculate the center frequency of the circuit.

$$\omega_0 = 316.23 \text{ rad/s}$$

The cutoff frequencies can be calculated using equations 9.5 and 9.6.

$$\omega_{c1} = 292.21 \text{ rad/s}$$

$$\omega_{c2} = 342.21 \text{ rad/s}$$

Calculate the bandwidth using equation 9.7.

$$\beta = 50 \text{ rad/s}$$

Use equation 9.8 to calculate the quality factor.

$$Q = 6.32$$

9.2.5 Determining Filter Type

To determine the type of filter a circuit corresponds to, first analyze the circuit given a DC steady-state ($\omega = 0$). In this case, all capacitors can be replaced with opens and all inductors can be replaced with shorts. Determine the value of the output voltage (which will either be zero or non-zero).

Second, analyze the circuit given a frequency approaching infinity ($\omega \rightarrow \infty$). In this case, all capacitors can be replaced with shorts and all inductors can be replaced with opens. Determine the value of the output

voltage (which will either be zero or non-zero).

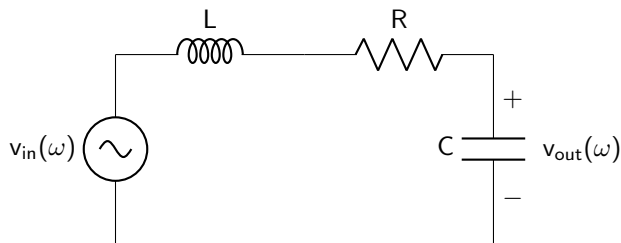
Table 9.1 can then be used to determine the filter type based on $v_{out}(0)$ and $v_{out}(\infty)$.

$v_{out}(0)$	$v_{out}(\infty)$	Filter Type
zero	zero	band-pass filter
zero	non-zero	high-pass filter
non-zero	zero	low-pass filter
non-zero	non-zero	band-stop filter

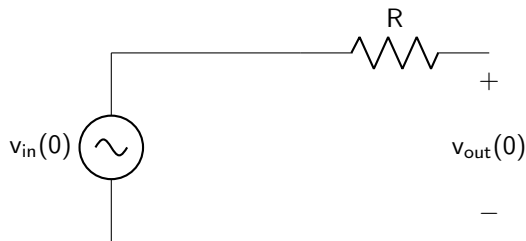
Table 9.1: It is possible to determine a filter type by analyzing the output voltage at low frequency ($\omega = 0$) and very high frequency ($\omega \rightarrow \infty$).

Example: Determining filter type

Determine the type of filter shown below.



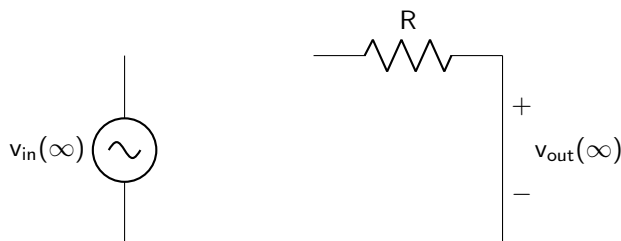
Re-draw the circuit for $\omega = 0$.



Determine the output voltage.

$$v_{out}(0) = v_{in}$$

Re-draw the circuit for $\omega \rightarrow \infty$.



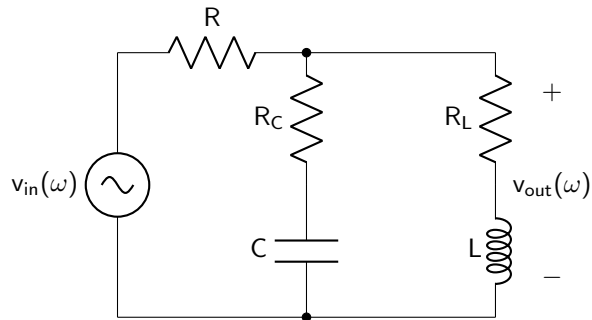
Determine the output voltage.

$$v_{out}(\infty) = 0$$

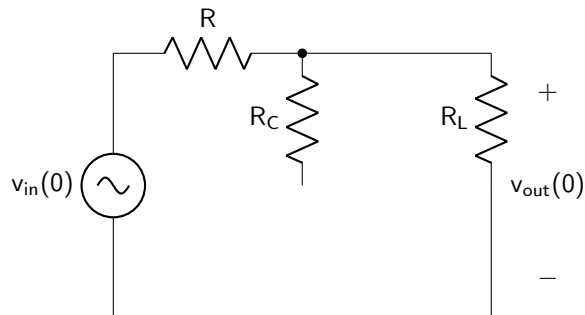
Consult table 9.1. The circuit is a low-pass filter.

Example: Determining filter type

Determine the type of filter shown below.



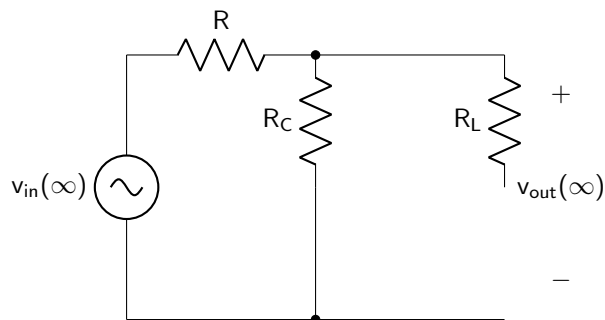
Re-draw the circuit for $\omega = 0$.



Determine the output voltage.

$$v_{out}(0) = v_{in} \frac{R_L}{R + R_L}$$

Re-draw the circuit for $\omega \rightarrow \infty$.



Determine the output voltage.

$$v_{out}(\infty) = v_{in} \frac{R_C}{R + R_C}$$

Consult table 9.1. The circuit is a band-stop filter.

9.3 Active Filters

It is possible to design filters that use an op-amp; these filters are known as active filters. Active filters are capable of generating output amplification (from the op-amp gain), and can also be used to create band-pass filters and band-stop filters without using inductors, which are physically large and contain parasitic resistance. While there are many different types of active filters, only a few will be discussed in this book.

9.3.1 First Order Active Filters

First order active filters are first order (one energy storage element) filters that use an op-amp. An active first order low-pass filter is shown in figure 9.22.

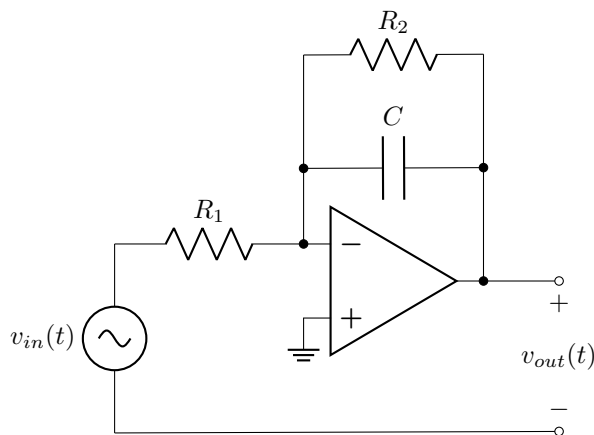


Figure 9.22: First order active low-pass filter.

The first order differential equation that describes the behavior of this low-pass filter is shown in equation 9.9, and can be derived by performing KCL at the inverting node of the op-amp.

$$-\frac{v_{in}(t)}{R_1 C} = \frac{dv_{out}(t)}{dt} + \frac{1}{R_2 C} v_{out}(t) \quad (9.9)$$

The cutoff frequency of the filter is defined by the coefficient on the $v_{out}(t)$ term, therefore the cutoff frequency of the first order active low-pass filter depicted in figure 9.22 is given in equation 9.10.

$$\omega_c = \frac{1}{R_2 C} \quad (9.10)$$

The gain of this active filter is given by the ratio of feedback resistor to input resistor, which can be seen

in the coefficient of the ratio of output to input voltages, shown in equation 9.11 and derived below. The initial equation in the derivation is obtained by performing KCL at the inverting input of the op-amp.

$$\frac{v_{out}(t)}{v_{in}(t)} = -\frac{R_2}{R_1} \left(\frac{\frac{1}{R_2 C}}{j\omega + \frac{1}{R_2 C}} \right) \quad (9.11)$$

DERIVATION

$$\begin{aligned} \frac{v_{in}(t)}{R_1} &= -\frac{v_{out}(t)}{R_2} - \frac{v_{out}(t)}{\frac{1}{j\omega C}} \\ &= -v_{out}(t) \left(\frac{1}{R_2} + j\omega C \right) \\ \frac{v_{out}(t)}{v_{in}(t)} &= -\frac{1}{R_1} \left(\frac{1}{\frac{1}{R_2} + j\omega C} \right) \\ &= -\frac{R_2}{R_1} \left(\frac{1}{1 + j\omega C R_2} \right) \\ &= -\frac{R_2}{R_1} \left(\frac{\frac{1}{R_2 C}}{j\omega + \frac{1}{R_2 C}} \right) \end{aligned}$$

An active first order high-pass filter is shown in figure 9.23.

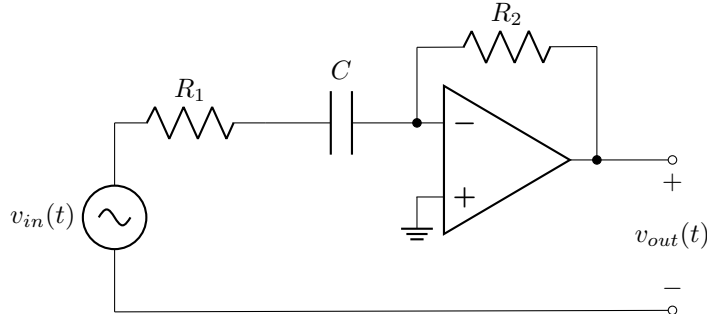


Figure 9.23: First order active high-pass filter.

The first order differential equation that describes the behavior of this high-pass filter is shown in equation 9.12, and can be derived by performing KCL at the inverting node of the op-amp.

$$-\frac{R_2}{R_1} \frac{dv_{in}(t)}{dt} = \frac{dv_{out}(t)}{dt} + \frac{1}{R_1 C} v_{out}(t) \quad (9.12)$$

The cutoff frequency of the filter is defined by the coefficient on the $v_{out}(t)$ term, therefore the cutoff frequency of the first order active high-pass filter depicted in figure 9.23 is given in equation 9.13.

$$\omega_c = \frac{1}{R_1 C} \quad (9.13)$$

The gain of this active filter is given by the ratio of feedback resistor to input resistor, which can be seen in the coefficient of the ratio of output to input voltages, shown in equation 9.14 and derived below. The initial equation in the derivation is obtained by performing KCL at the inverting input of the op-amp.

$$\frac{v_{out}(t)}{v_{in}(t)} = -\frac{R_2}{R_1} \left(\frac{j\omega}{j\omega + \frac{1}{R_1 C}} \right) \quad (9.14)$$

DERIVATION

$$\begin{aligned} \frac{v_{in}(t)}{R_1 + \frac{1}{j\omega C}} &= -\frac{v_{out}(t)}{R_2} \\ \frac{v_{out}(t)}{v_{in}(t)} &= -\frac{R_2}{R_1 + \frac{1}{j\omega C}} \\ &= -\frac{j\omega C R_2}{j\omega C R_1 + 1} \\ &= -\frac{R_2}{R_1} \left(\frac{j\omega}{j\omega + \frac{1}{R_1 C}} \right) \end{aligned}$$

9.3.2 Cascaded Filters

As mentioned in section 4.5.3, op-amps can be cascaded together. While section 4.5.3 made reference to the product of each individual circuit gain, cascaded filters can be analyzed by calculating the product of each filter's transfer function (discussed in section 9.4). In this manner, band-pass filters, band-stop filters, and second order (or higher) high-pass and low-pass filters can be designed.

A low-pass filter whose input is fed into a high-pass filter acts as a band-pass filter. A cascaded active band-pass filter is depicted in figure 9.24.

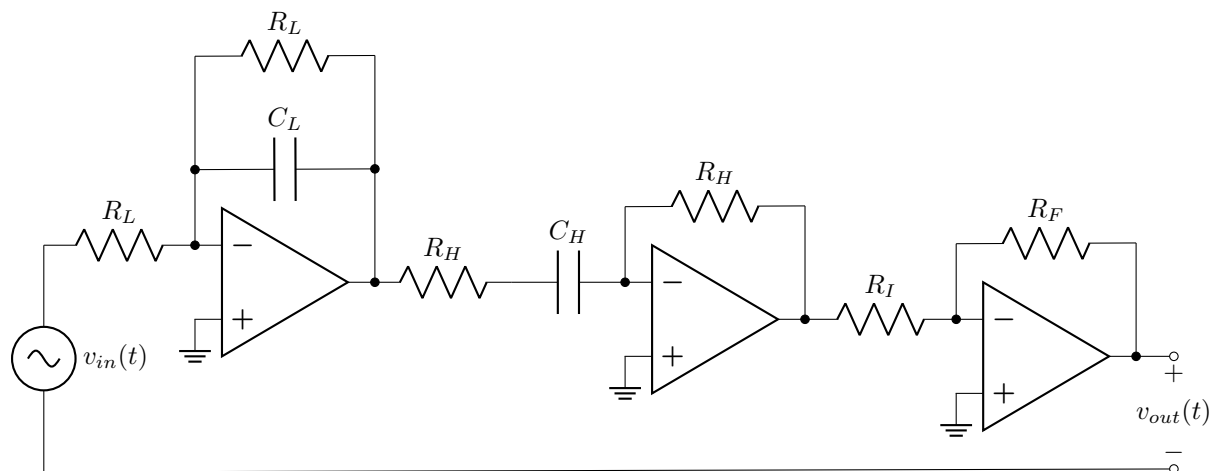


Figure 9.24: Cascaded active band-pass filter.

This filter has three stages. The first stage is a low-pass filter, the second stage is a high-pass filter, and the third stage is an inverting amplifier to provide overall amplification (gain). The low-pass filter stage sets the upper cutoff frequency, which was defined in equation 9.6 as ω_{c2} . Using equation 9.10, we can derive the upper cutoff frequency as shown in equation 9.15.

$$\omega_{c2} = \frac{1}{R_L C_L} \quad (9.15)$$

The high-pass filter stage sets the lower cutoff frequency, which was defined in equation 9.5 as ω_{c1} . Using equation 9.13, we can derive the lower cutoff frequency as shown in equation 9.16.

$$\omega_{c1} = \frac{1}{R_H C_H} \quad (9.16)$$

The final stage has a gain given by equation 4.9 and sets the overall gain of the band-pass filter circuit. In this manner, each of the circuit properties can be tuned in each individual stage.

A low-pass filter and high-pass filter can also be used as inputs to a summing amplifier, which leads to an active band-stop filter. This circuit is depicted in figure 9.25.

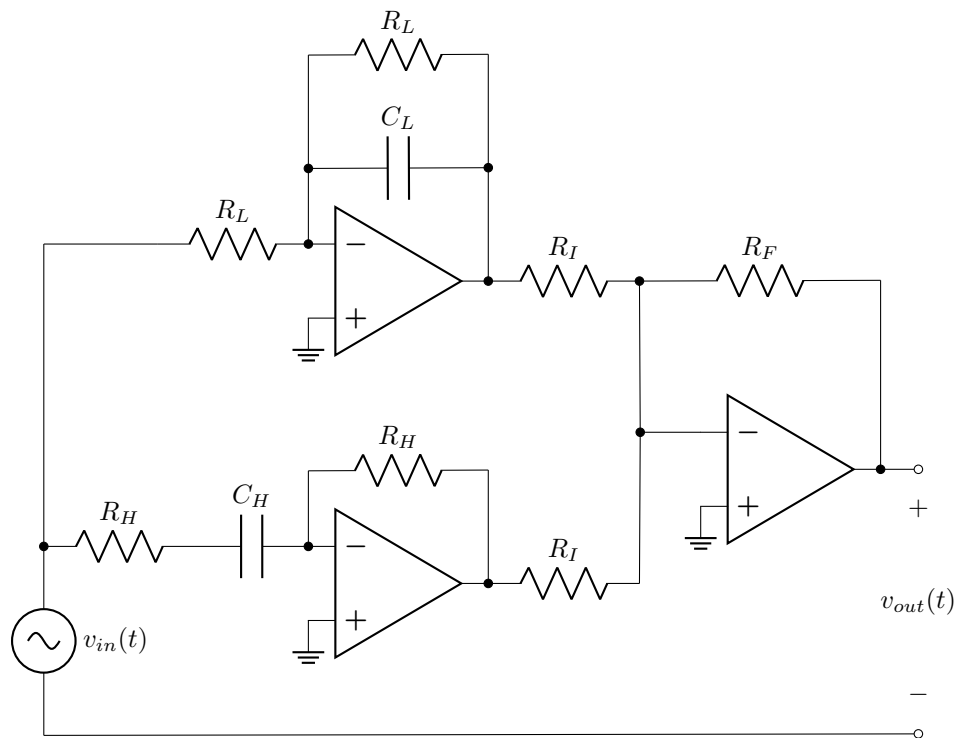


Figure 9.25: Cascaded active band-stop filter.

The low-pass filter sets the lower cutoff frequency of the stopband and the high-pass filter sets the upper cutoff frequency of the stopband. The summing amplifier sets the overall gain of the band-stop filter circuit.

This gain is defined by the input and feedback resistors, described in equation 9.17.

$$A = -\frac{R_F}{R_I} \quad (9.17)$$

Finally, higher order active filters can be created by cascading together multiple low-pass filters or multiple high-pass filters (for example). Cascading together identical filters creates a sharper slope at the cutoff frequency. An inverting amplifier can be staged at the output to create an overall circuit gain.

9.3.3 Other Active Filter Topologies

While there are numerous other active (and passive) filter designs that will not be explored in this book, one other active filter topology will be described. This is called the Sallen-Key topology. A low-pass Sallen-Key filter is depicted in figure 9.26. (Swapping all resistors for capacitors and vice versa will result in a high-pass Sallen-Key filter. A band-pass Sallen-Key filter can also be realized.)

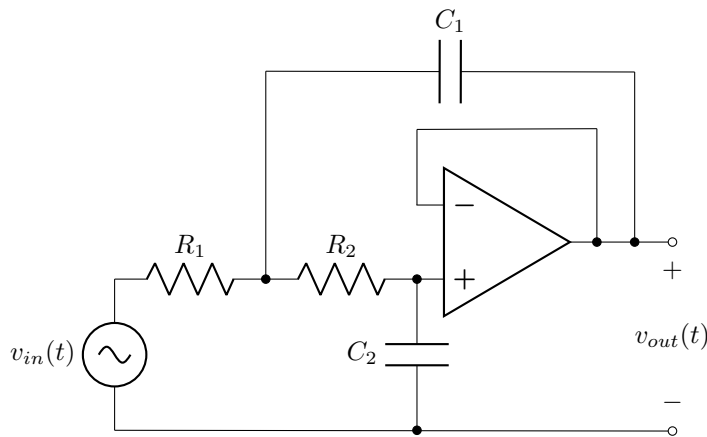


Figure 9.26: A unity gain Sallen-Key low-pass filter circuit.

The voltage follower configuration of the circuit gives it a gain of one (known as unity gain), and ensures that the voltage at the non-inverting input of the op-amp is equal to $v_{out}(t)$. If a filter gain is desired, then the voltage follower configuration can be altered to include divided feedback instead of shorting the inverting input to the output.

The Sallen-Key filter shown in figure 9.26 has a differential equation described in equation 9.18 and is derived below using KCL at the nodes connecting the two resistors to capacitor C_1 and also at the non-inverting input of the op-amp. (The term $v_a(t)$ in the derivation denotes the node connecting the two resistors to capacitor C_1 .)

$$\frac{1}{R_1 R_2 C_1 C_2} v_{in}(t) = \frac{d^2 v_{out}(t)}{dt^2} + \left[\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} \right] \frac{d v_{out}(t)}{dt} + \left[\frac{1}{R_1 R_2 C_1 C_2} \right] v_{out}(t) \quad (9.18)$$

DERIVATION

$$\begin{aligned}
\frac{v_{in}(t) - v_a(t)}{R_1} &= C_1 \frac{d}{dt} [v_a(t) - v_{out}(t)] + C_2 \frac{dv_{out}(t)}{dt} \\
\frac{v_{in}(t)}{R_1} &= \left[\frac{1}{R_1} + C_1 \frac{d}{dt} \right] v_a(t) + \left[C_2 \frac{d}{dt} - C_1 \frac{d}{dt} \right] v_{out}(t) \\
\frac{v_a(t) - v_{out}(t)}{R_2} &= C_2 \frac{dv_{out}(t)}{dt} \\
v_a(t) &= R_2 C_2 \frac{dv_{out}(t)}{dt} + v_{out}(t) \\
\frac{v_{in}(t)}{R_1} &= \left[\frac{1}{R_1} + C_1 \frac{d}{dt} \right] \left[R_2 C_2 \frac{dv_{out}(t)}{dt} + v_{out}(t) \right] + \left[C_2 \frac{d}{dt} - C_1 \frac{d}{dt} \right] v_{out}(t) \\
&= \frac{R_2 C_2}{R_1} \frac{dv_{out}(t)}{dt} + \frac{1}{R_1} v_{out}(t) + C_1 R_2 C_2 \frac{d^2 v_{out}(t)}{dt^2} + C_2 \frac{dv_{out}(t)}{dt} \\
&= R_2 C_1 C_2 \frac{d^2 v_{out}(t)}{dt^2} + \left[\frac{R_2 C_2}{R_1} + C_2 \right] \frac{dv_{out}(t)}{dt} + \left[\frac{1}{R_1} \right] v_{out}(t) \\
\frac{1}{R_1 R_2 C_1 C_2} v_{in}(t) &= \frac{d^2 v_{out}(t)}{dt^2} + \left[\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} \right] \frac{dv_{out}(t)}{dt} + \left[\frac{1}{R_1 R_2 C_1 C_2} \right] v_{out}(t)
\end{aligned}$$

9.4 Transfer Functions

All of the analysis that was carried out on filters in the previous sections of this book looked at the magnitude response of $v_{out}(t)/v_{in}(t)$ at different frequencies. A transfer function is defined as the frequency response of a system, described in equation 9.19.

$$H(j\omega) = \frac{v_{out}(j\omega)}{v_{in}(j\omega)} \quad (9.19)$$

The transfer function provides a lot of useful information about the response of a circuit across many different frequencies of operation. The transfer function is a complex function of $j\omega$. It can be split into magnitude and phase responses. The magnitude of the transfer function is calculated by taking the absolute value of $H(j\omega)$, as shown in equation 9.20.

$$|H(j\omega)| = \sqrt{(\Re H(j\omega))^2 + (\Im H(j\omega))^2} \quad (9.20)$$

The phase response is calculated by finding the angle between the real and imaginary parts, as shown in equation 9.21.

$$\angle H(j\omega) = \text{atan2}(\Im H(j\omega), \Re H(j\omega)) \quad (9.21)$$

When plotted together, this is called a Bode plot. A Bode plot shows the frequency response of a system by showing the magnitude and phase response of the system.

The transfer function of a circuit is derived by using a phasor transformed circuit to calculate the output voltage divided by the input voltage as a function of $j\omega$. The function is normalized by forcing the highest order term of $j\omega$ (which, for a second order circuit, is $-\omega^2$) to have a coefficient of one.

Consider the phasor transformed low-pass filter circuit shown in figure 9.27.

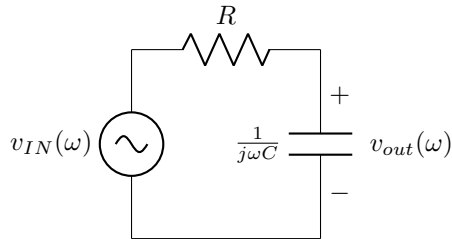


Figure 9.27: A phasor transformed low-pass filter circuit.

The voltage divider rule can be used to determine the transfer function of the circuit, shown in equation 9.22 and derived below.

$$H(j\omega) = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}} \quad (9.22)$$

DERIVATION

$$\begin{aligned} H(j\omega) &= \left(\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \right) \\ &= \frac{1}{j\omega RC + 1} \\ &= \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}} \end{aligned}$$

The magnitude response of the filter is shown in equation 9.23 and derived below.

$$|H(j\omega)| = \frac{\frac{1}{RC}}{\sqrt{-\omega^2 + \left(\frac{1}{RC}\right)^2}} \quad (9.23)$$

DERIVATION

$$\begin{aligned} |H(j\omega)| &= \left| \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}} \right| \\ &= \frac{\left| \frac{1}{RC} \right|}{\left| j\omega + \frac{1}{RC} \right|} \\ &= \frac{\frac{1}{RC}}{\sqrt{-\omega^2 + \left(\frac{1}{RC}\right)^2}} \end{aligned}$$

The phase response of the filter is shown in equation 9.24 and derived below.

$$\angle H(j\omega) = -\text{atan2}\left(\omega, \frac{1}{RC}\right) \quad (9.24)$$

DERIVATION

$$\begin{aligned} \angle H(j\omega) &= \frac{\angle\left(\frac{1}{RC}\right)}{\angle\left(j\omega + \frac{1}{RC}\right)} \\ &= \frac{0^\circ}{\text{atan2}\left(\omega, \frac{1}{RC}\right)} \\ &= -\text{atan2}\left(\omega, \frac{1}{RC}\right) \end{aligned}$$

Given values of $R = 100 \, \Omega$ and $C = 100 \, \mu\text{F}$, it is possible to create a Bode plot of the filter. This is shown in figure 9.28.

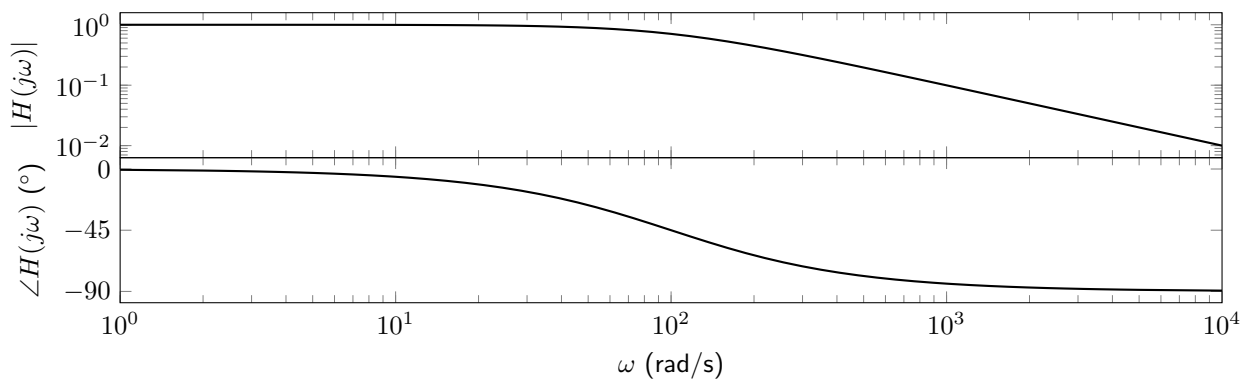
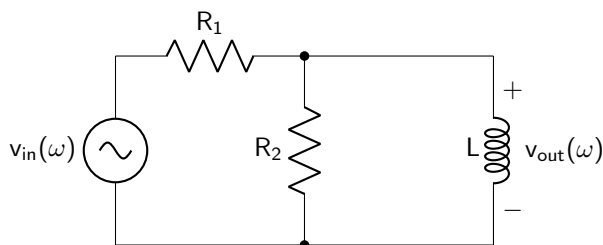


Figure 9.28: Bode plot for the low-pass filter circuit shown in figure 9.27 given $R = 100 \, \Omega$ and $C = 100 \, \mu\text{F}$.

Example: Transfer function in symbolic form

Derive a transfer function for the circuit shown below. Then, determine the filter type.



This transfer function can be derived using a voltage divider. Combine the impedances of resistor R_2

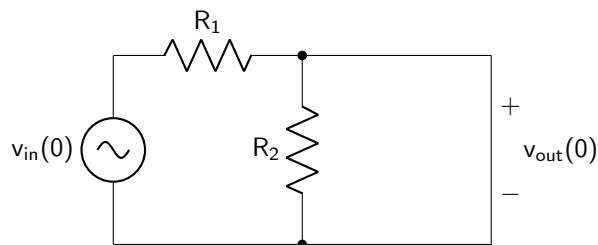
and the inductor in parallel.

$$Z_{RL} = \frac{j\omega LR_2}{R_2 + j\omega L}$$

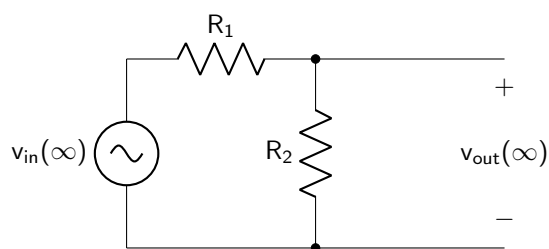
Apply the voltage divider rule, and minimize until the transfer function is normalized.

$$\begin{aligned} v_{out}(\omega) &= v_{in}(\omega) \left[\frac{\frac{j\omega LR_2}{R_2 + j\omega L}}{R_1 + \frac{j\omega LR_2}{R_2 + j\omega L}} \right] \\ H(j\omega) &= \frac{\frac{j\omega LR_2}{R_2 + j\omega L}}{R_1 + \frac{j\omega LR_2}{R_2 + j\omega L}} \\ &= \frac{j\omega LR_2}{j\omega LR_1 + R_1 R_2 + j\omega LR_2} \\ &= \frac{j\omega LR_2}{j\omega L(R_1 + R_2) + R_1 R_2} \\ &= \frac{j\omega \frac{R_2}{R_1 + R_2}}{j\omega + \frac{R_1 R_2}{L(R_1 + R_2)}} \end{aligned}$$

To determine the filter type, first analyze the circuit for $\omega = 0$.



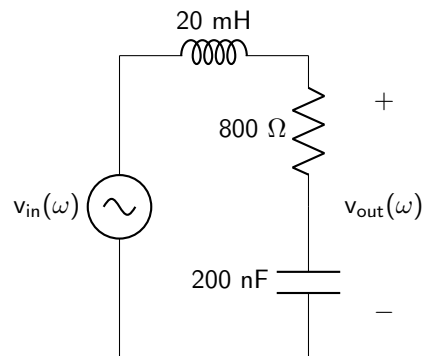
The output voltage will be zero at $\omega = 0$ due to the inductor being a short in those conditions. Then, analyze the circuit for $\omega \rightarrow \infty$.



The output voltage will be non-zero at $\omega = \infty$ due to the inductor being an open in those conditions. Therefore, this is a high-pass filter.

Example: Transfer function in numeric form

Derive a transfer function for the circuit shown below. Then, determine the filter type.



It is advantageous to begin by deriving the transfer function in symbolic form, and then substituting numeric values in at the end. This circuit's transfer function can be derived by applying the voltage divider rule and minimizing until the transfer function is normalized.

$$v_{out}(\omega) = v_{in}(\omega) \left[\frac{R + \frac{1}{j\omega C}}{j\omega L + R + \frac{1}{j\omega C}} \right]$$

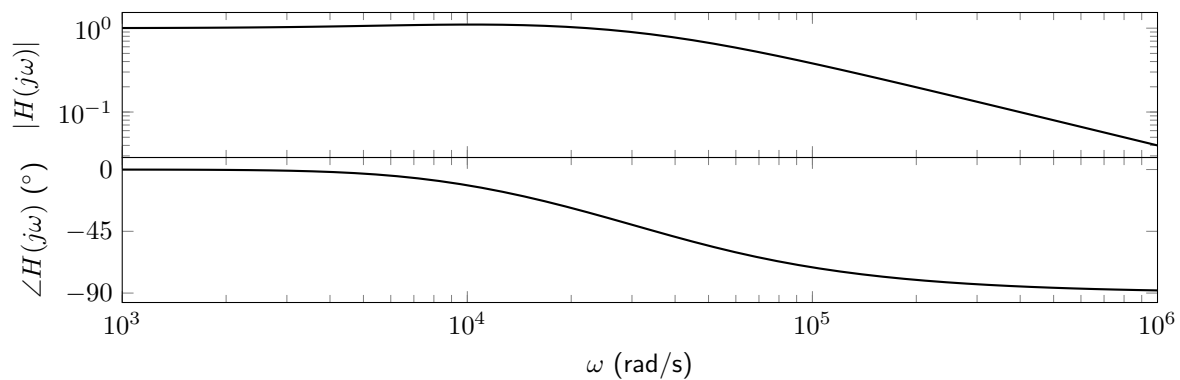
$$H(j\omega) = \frac{j\omega CR + 1}{-\omega^2 LC + j\omega CR + 1}$$

$$= \frac{j\omega \frac{R}{L} + \frac{1}{LC}}{-\omega^2 + j\omega \frac{R}{L} + \frac{1}{LC}}$$

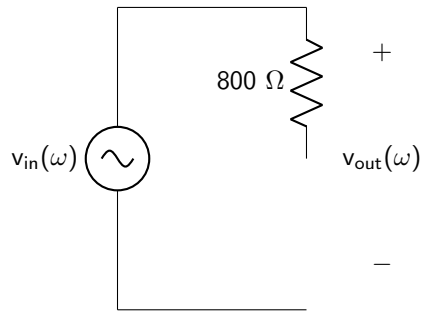
Now that the transfer function has been derived symbolically, a numerical solution can be obtained.

$$H(j\omega) = \frac{j\omega 40000 + 2.5 \times 10^8}{-\omega^2 + j\omega 40000 + 2.5 \times 10^8}$$

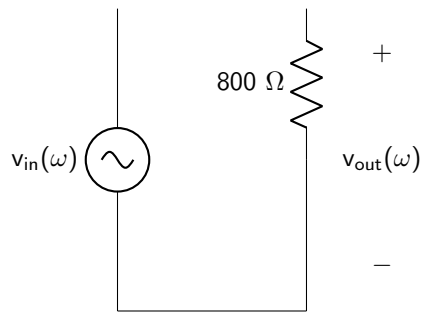
The bode plot for this filter is shown below.



To determine the filter type, first analyze the circuit for $\omega = 0$.



The output voltage will be non-zero at $\omega = 0$. Then, analyze the circuit for $\omega \rightarrow \infty$.



The output voltage will be zero at $\omega = \infty$. Therefore, this is a low-pass filter. This analysis is verified by the Bode plot.

9.4.1 Relating Transfer Functions and Differential Equations

After close observation, it can be seen that a transfer function contains exactly the same information as the differential equation that describes a circuit. Consider the circuit in figure 9.27. KCL can be used to derive the differential equation, shown in equation 9.25.

$$\frac{1}{RC}v_{in}(t) = \frac{d}{dt}v_{out}(t) + \frac{1}{RC}v_{out}(t) \quad (9.25)$$

It is possible to replace every d/dt term with $j\omega$, every t term with ω , and solve for v_{out}/v_{in} . This is shown in equation 9.26 and derived below.

$$\frac{v_{out}(\omega)}{v_{in}(\omega)} = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}} \quad (9.26)$$

DERIVATION

$$\begin{aligned}\frac{1}{RC}v_{in}(t) &= j\omega v_{out}(t) + \frac{1}{RC}v_{out}(t) \\ \frac{1}{RC}v_{in}(\omega) &= v_{out}(\omega) \left(j\omega + \frac{1}{RC} \right) \\ \frac{v_{out}(\omega)}{v_{in}(\omega)} &= \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}\end{aligned}$$

Note that this is exactly the same as the transfer function derived in equation 9.22!

9.4.2 Understanding Transfer Function Poles and Zeros

Identifying the poles and zeros of a transfer function can provide information about the stability of a system. Poles are the roots of the denominator of the transfer function; zeros are the roots of the numerator of the transfer function. Stability relates to what happens after a system is disturbed from a condition of equilibrium. If, after a disturbance, a system tends to return to equilibrium, that system is said to be stable. If, after a disturbance, the system remains in the new state without changing, that system is said to be marginally stable. If, after a disturbance, the system diverges away from equilibrium, that system is said to be unstable.

In resistive circuits (such as the ones described in this book), the systems are stable due to the damping property of resistors. This causes all exponential terms in output voltage equations to have negative powers, causing them to decay over time. All poles have negative real values. The greater the magnitude of the real part of the pole, the faster the exponential term will decay.

Second order systems (such as second order circuits) will have two poles. If the two poles have real and distinct negative values, then the system will have an overdamped transient response. If the two poles have real and repeated negative values, then the system will have a critically damped transient response. If the two poles are complex conjugate pairs, then the system will have an underdamped transient response.

Example: Relating poles to the transient response (overdamped)

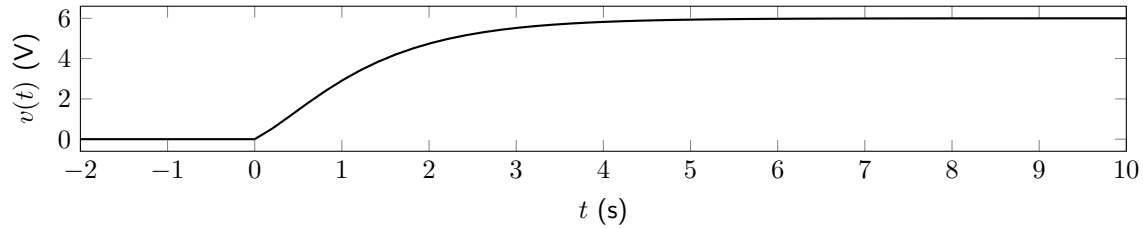
A second order circuit has a transfer function described below.

$$H(j\omega) = \frac{j\omega + 6}{-\omega^2 + j\omega 3 + 2}$$

There is one zero at -6 , and two poles (at -2 and -1). The transfer function can be rewritten.

$$H(j\omega) = \frac{(j\omega + 6)}{(j\omega + 2)(j\omega + 1)}$$

Because the poles are real and distinct, the transient response of this circuit will be overdamped. The step transient response of the circuit is plotted below as a function of time.



Example: Relating poles to the transient response (critically damped)

A second order circuit has a transfer function described below.

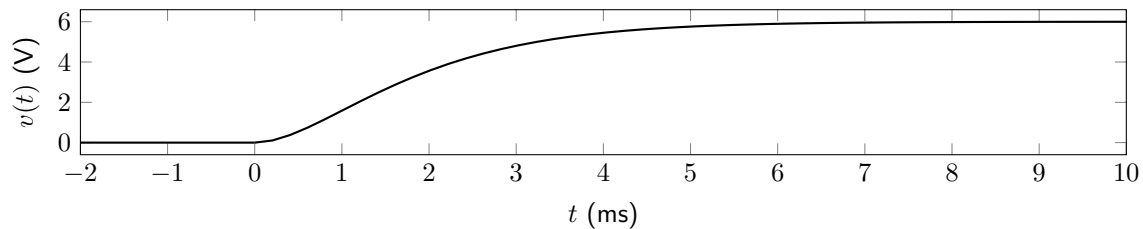
$$H(j\omega) = \frac{1}{-\omega^2 + j\omega 2000 + 1 \times 10^6}$$

There are no zeros, and two repeated poles at -1000 . The transfer function can be rewritten.

$$H(j\omega) = \frac{1}{(j\omega + 1000)(j\omega + 1000)}$$

Because the poles are real and repeated, the transient response of this circuit will be critically damped.

The step transient response of the circuit is plotted below as a function of time.



Example: Relating poles to the transient response (underdamped)

A second order circuit has a transfer function described below.

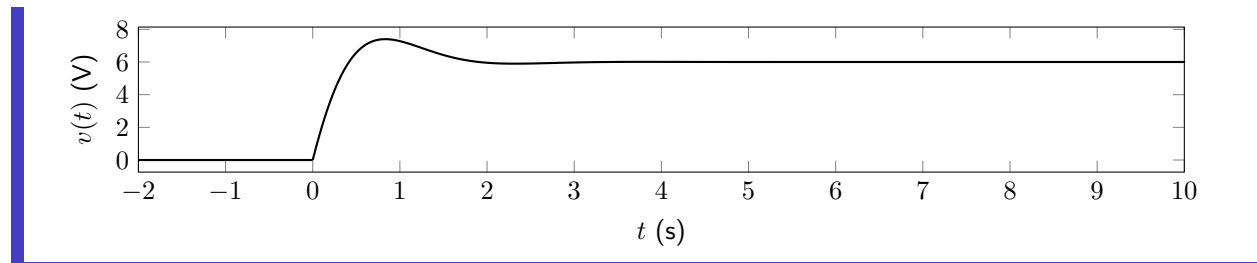
$$H(j\omega) = \frac{-\omega^2}{-\omega^2 + j\omega 3.5 + 7.5}$$

There are no zeros, and two repeated poles at -1000 . The transfer function can be rewritten.

$$H(j\omega) = \frac{(j\omega + 0)(j\omega + 0)}{(j\omega + 1.75 + j\sqrt{4.4375})(j\omega + 1.75 - j\sqrt{4.4375})}$$

Because the poles are complex conjugate pairs, the transient response of this circuit will be underdamped.

The step transient response of the circuit is plotted below as a function of time.



A transfer function can be re-written as shown in equation 9.27. Each of the z values is a zero, and each of the p values is a pole. A first order system (such as a first order circuit) will have one pole. A second order system will have two poles.

$$H(j\omega) = \frac{(j\omega - z_1)(j\omega - z_2)\dots(j\omega - z_n)}{(j\omega - p_1)(j\omega - p_2)\dots(j\omega - p_n)} \quad (9.27)$$

Example: Poles and zeros

Determine the poles and zeros of the transfer function of the second order low-pass filter from figure 9.9, where $R_1 = 10 \Omega$, $R_2 = 20 \Omega$, $C_1 = 200 \mu\text{F}$, and $C_2 = 500 \mu\text{F}$.

The transfer function of the circuit is given below.

$$H(j\omega) = \frac{50000}{-\omega^2 + j\omega 850 + 50000}$$

The transfer function has no zeros and two poles (at -786.42 and -63.58). The transfer function can be rewritten.

$$H(j\omega) = \frac{50000}{(j\omega + 786.42)(j\omega + 63.58)}$$

Poles and zeros can be plotted on the complex plane. Each pole is depicted with an \times symbol, and each zero is depicted with a \circ . Based on the location of the poles in the complex plane, the stability of the system can be quickly determined. Pole-zero diagrams for stable second order systems are depicted in figure 9.29.

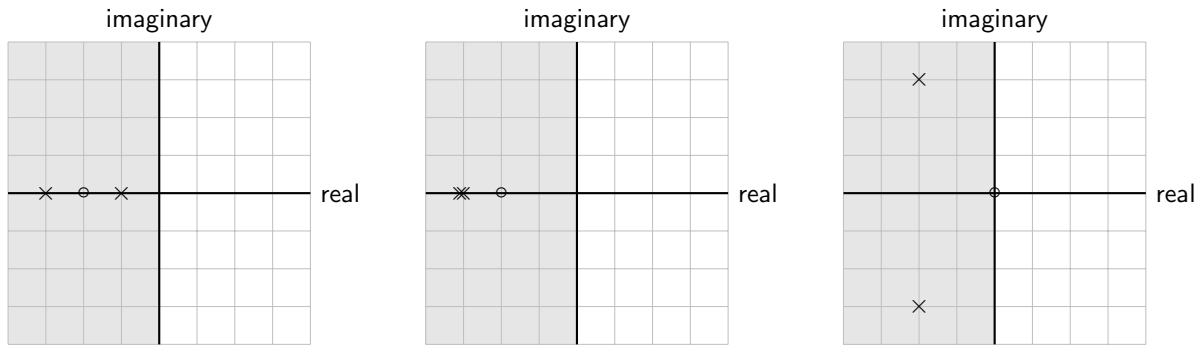


Figure 9.29: Distinct poles lead to an underdamped response (left). Repeated poles lead to a critically damped response (middle). Complex conjugate poles lead to an underdamped response (right). All three of these pole-zero diagrams is stable.

Unstable systems have increasing exponential functions in the output response. In the real physical world, this output response is impossible. Not only is there only a finite amount of energy (which means that no output, whether it be voltage, current, or a non-electrical property, can increase infinitely), but any physical system will eventually break once the output response reaches a certain value. An unstable second order system with distinct or repeated real roots is depicted in figure 9.30.

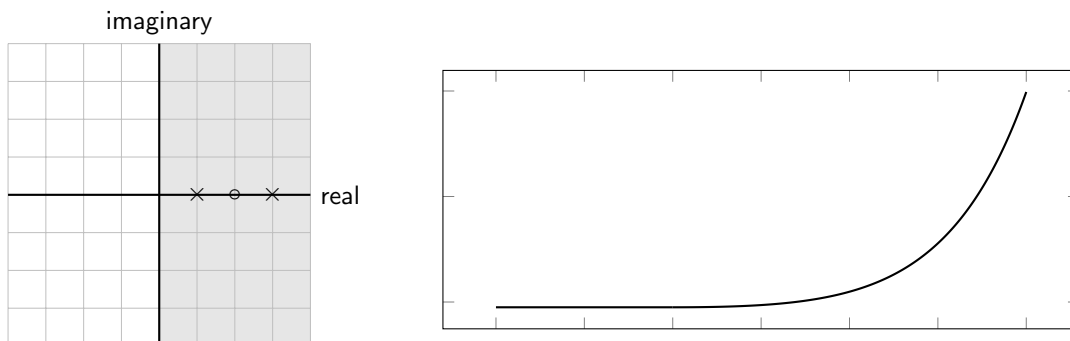


Figure 9.30: Poles with positive real parts (whether they are distinct, as shown, or repeated, not shown) lead to an unstable response (left). The system output consists of increasing exponentials (right).

An unstable system with complex conjugate roots (with positive real parts) has an output response of an increasing amplitude sinusoid. This is depicted in figure 9.31.

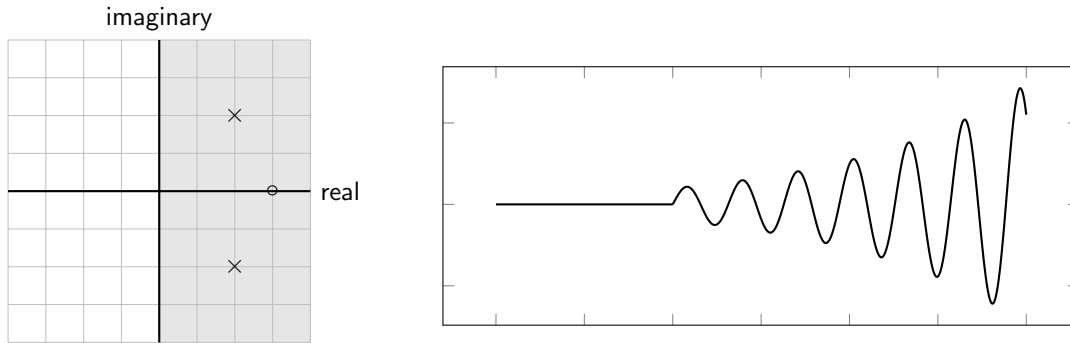


Figure 9.31: Poles that are complex conjugates with positive real parts lead to an unstable response (left). The system output consists of an increasing amplitude sinusoid (right).

Marginally stable systems have constant amplitude functions in the output response. A second order system with two poles of zero is depicted in figure 9.32.

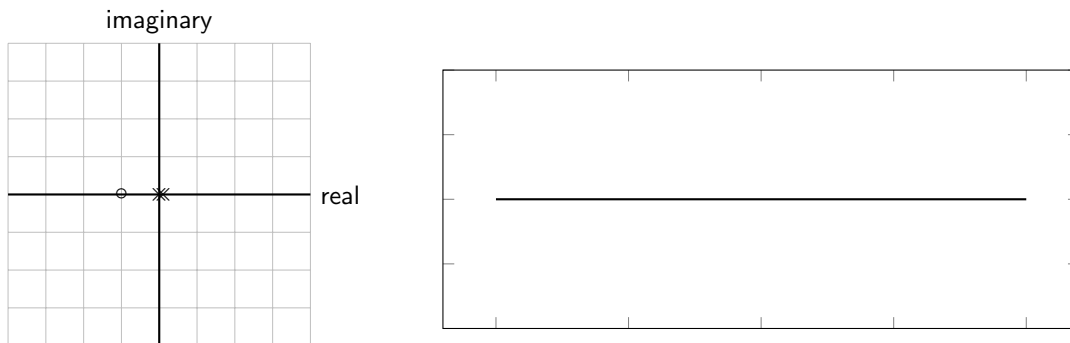


Figure 9.32: Poles equal to zero lead to a marginally stable response (left). The system output consists of a constant value (right).

A marginally stable system with complex conjugate pair poles (with no real part) has a constant amplitude sinusoid as a response. This would be the case for an ideal LC circuit with no parasitic resistance. The output response is depicted in figure 9.33.

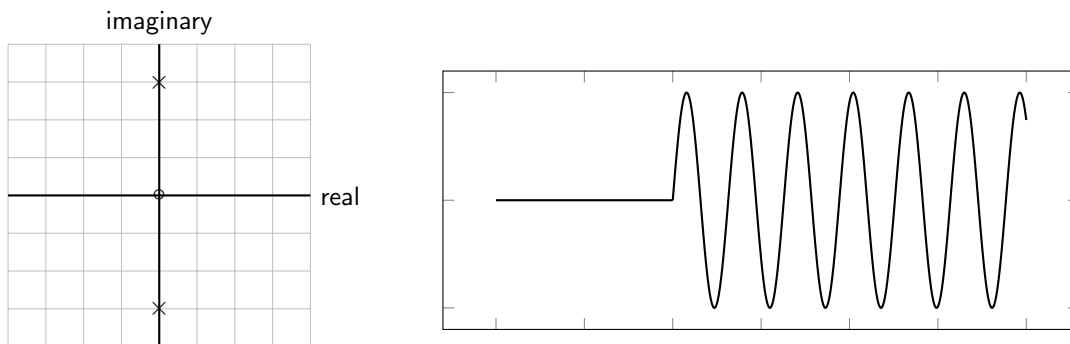


Figure 9.33: Poles that are complex conjugates with zero real parts lead to a marginally stable response (left). The system output consists of a constant amplitude sinusoid (right).

Extra Resources

- Cheever, Erik. Bode Plots Overview. Retrieved 3 March 2021. <https://lpsa.swarthmore.edu/Bode/Bode.html>
- Massachusetts Institute of Technology Department of Mechanical Engineering. Retrieved 2 September 2021. <https://web.mit.edu/2.14/www/Handouts/PoleZero.pdf>
- National Semiconductor Corporation, 2010. Application Note 779: A Basic Introduction to Filters - Active, Passive, and Switched Capacitor. <https://www.ti.com/lit/an/snoa224a/snoa224a.pdf>
- Texas Instruments, 2002. Application Report: Analysis of the Sallen-Key Architecture. <https://www.ti.com/lit/an/sloa024b/sloa024b.pdf>

Example Problems

Complex Voltage and Current Divider

1. Calculate $v(t)$ given the circuit diagram shown in figure 9.34.

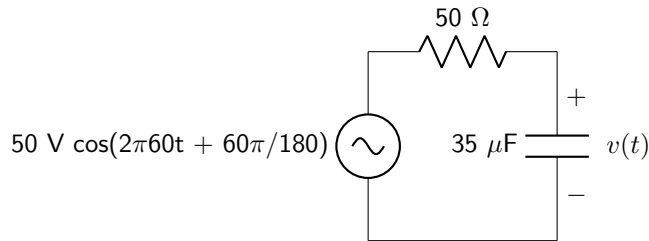


Figure 9.34: Circuit diagram for complex voltage and current divider circuits question 1.

2. Calculate $v(t)$ given the circuit diagram shown in figure 9.35. The frequency of operation is 80 Hz.

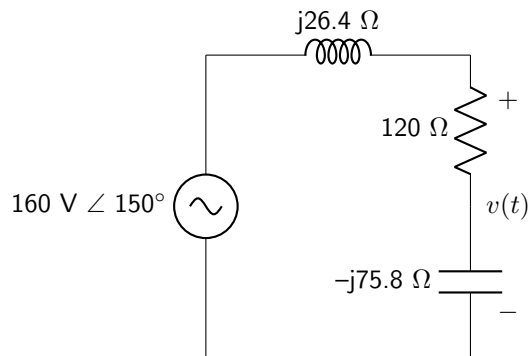


Figure 9.35: Circuit diagram for complex voltage and current divider circuits question 2.

3. Calculate $i(t)$ given the circuit diagram shown in figure 9.36. The frequency of operation is 150 Hz.

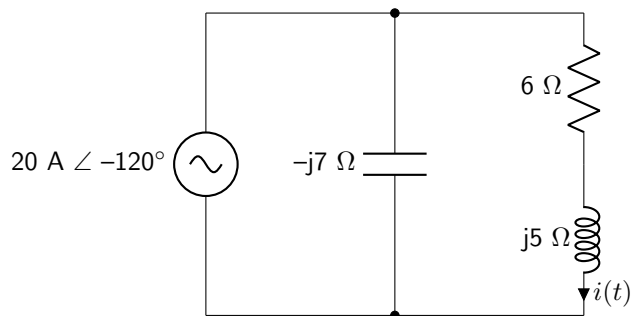


Figure 9.36: Circuit diagram for complex voltage and current divider circuits question 3.

4. Calculate $v(t)$ given the circuit diagram shown in figure 9.37.

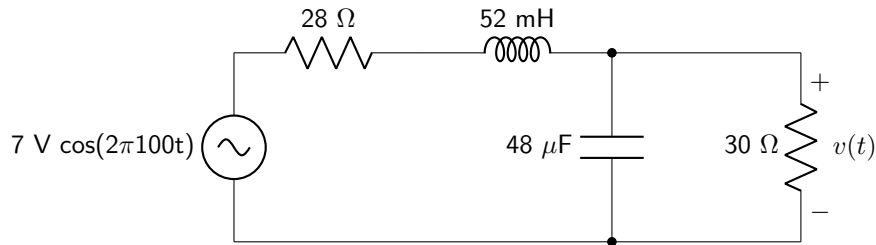


Figure 9.37: Circuit diagram for complex voltage and current divider circuits question 4.

5. Calculate $i(t)$ given the circuit diagram shown in figure 9.38.

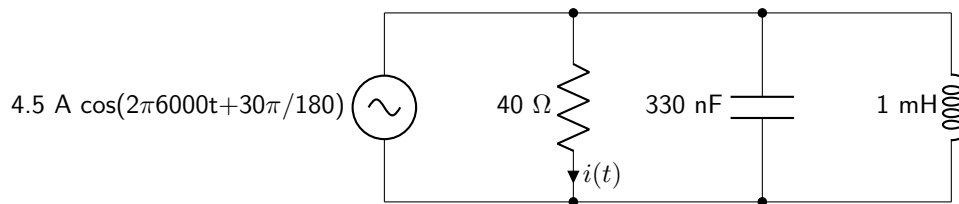


Figure 9.38: Circuit diagram for complex voltage and current divider circuits question 5.

Complex Kirchhoff's Laws

6. Calculate $i(t)$ given the circuit diagram shown in figure 9.39.

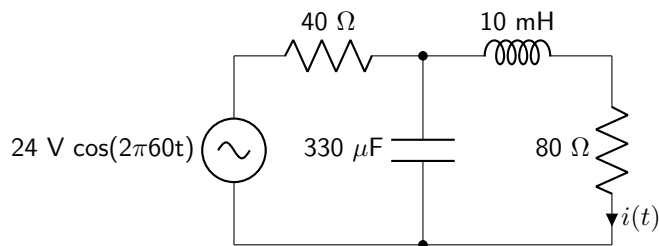


Figure 9.39: Circuit diagram for complex Kirchhoff's laws question 6.

7. Calculate $v(t)$ given the circuit diagram shown in figure 9.40. The frequency of operation is 20 Hz.

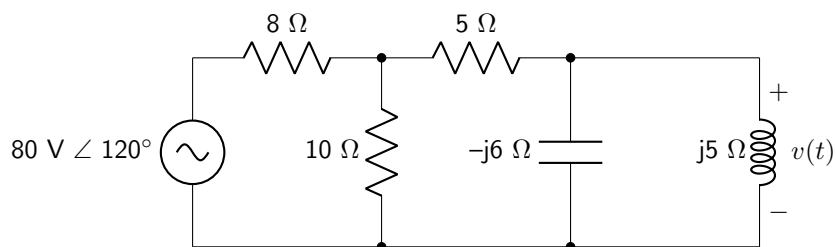


Figure 9.40: Circuit diagram for complex Kirchhoff's laws question 7.

8. Calculate $v(t)$ given the circuit diagram shown in figure 9.41. The frequency of operation is 20 kHz.

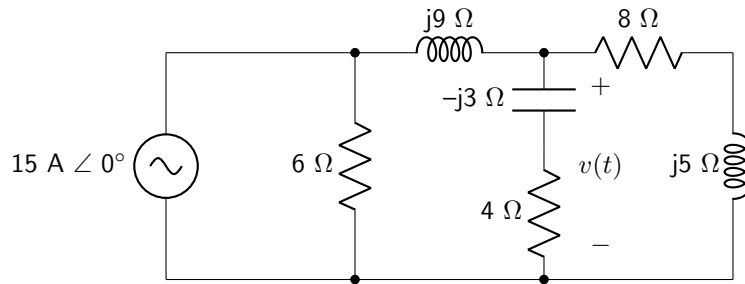


Figure 9.41: Circuit diagram for complex Kirchhoff's laws question 8.

9. Calculate $v(t)$ given the circuit diagram shown in figure 9.42. The frequency of operation is 30 Hz.

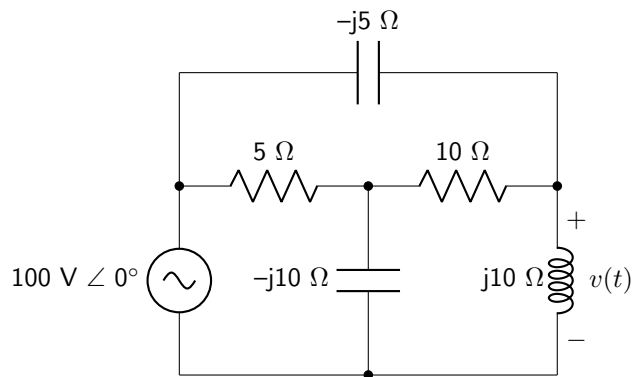


Figure 9.42: Circuit diagram for complex Kirchhoff's laws question 9.

10. Calculate $v(t)$ given the circuit diagram shown in figure 9.43.

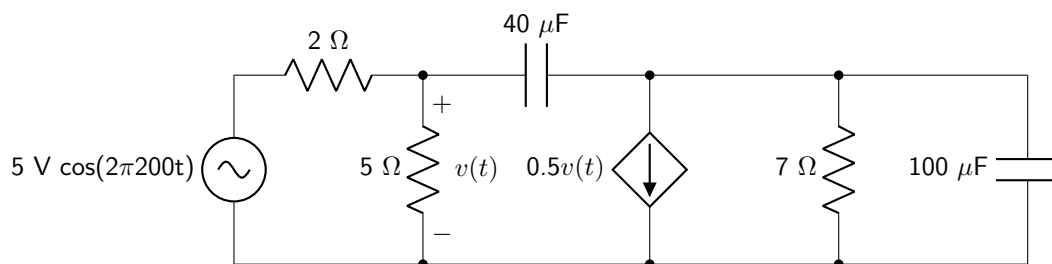


Figure 9.43: Circuit diagram for complex Kirchhoff's laws question 10.

Complex Mesh Analysis

11. Calculate $v(t)$ given the circuit diagram shown in figure 9.44. The frequency of operation is 50 Hz.

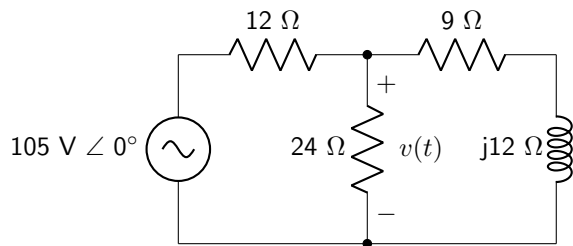


Figure 9.44: Circuit diagram for complex mesh analysis question 11.

12. Calculate $i(t)$ given the circuit diagram shown in figure 9.45.

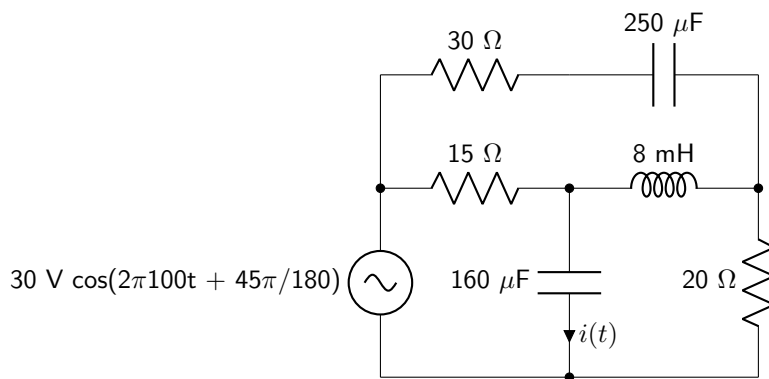


Figure 9.45: Circuit diagram for complex mesh analysis question 12.

13. Calculate $v(t)$ given the circuit diagram shown in figure 9.46.

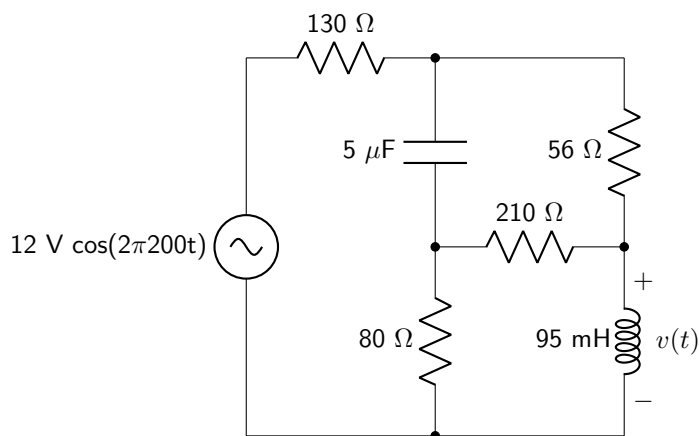


Figure 9.46: Circuit diagram for complex mesh analysis question 13.

14. Calculate $v(t)$ given the circuit diagram shown in figure 9.47. The frequency of operation is 5 kHz.

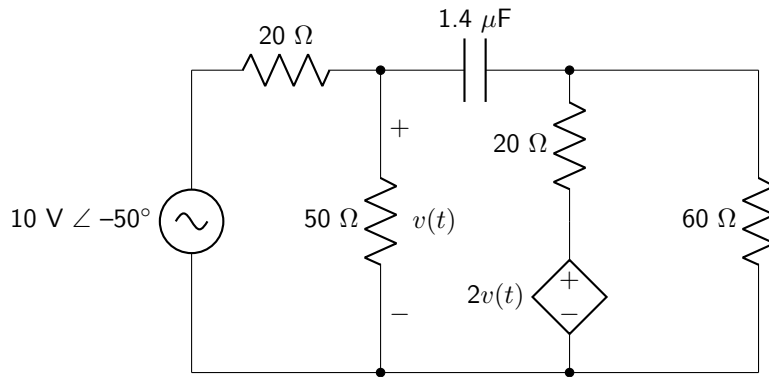


Figure 9.47: Circuit diagram for complex mesh analysis question 14.

15. Calculate $v(t)$ given the circuit diagram shown in figure 9.48.

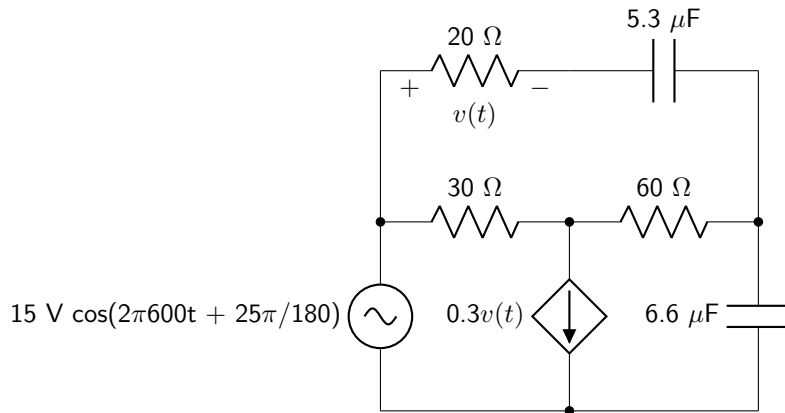


Figure 9.48: Circuit diagram for complex mesh analysis question 15.

Complex Superposition

16. Calculate $v(t)$ given the circuit diagram shown in figure 9.49. The frequency of operation is 10 kHz.

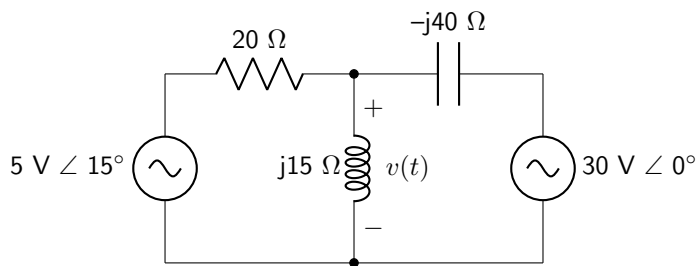


Figure 9.49: Circuit diagram for complex superposition question 16.

17. Calculate $v(t)$ given the circuit diagram shown in figure 9.50. The frequency of operation is 40 Hz.

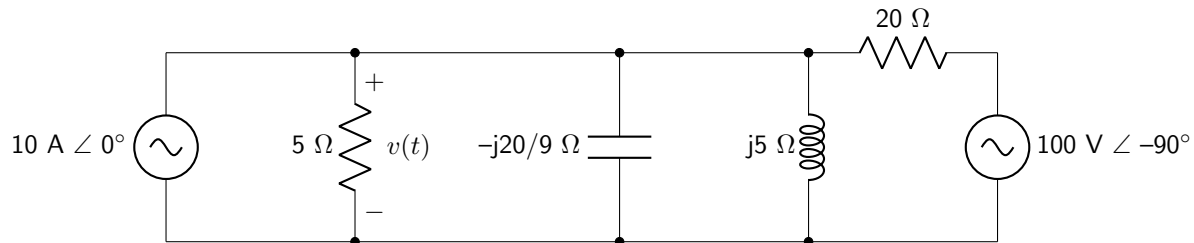


Figure 9.50: Circuit diagram for complex superposition question 17.

18. Calculate $v(t)$ given the circuit diagram shown in figure 9.51. The frequency of operation is 100 Hz.

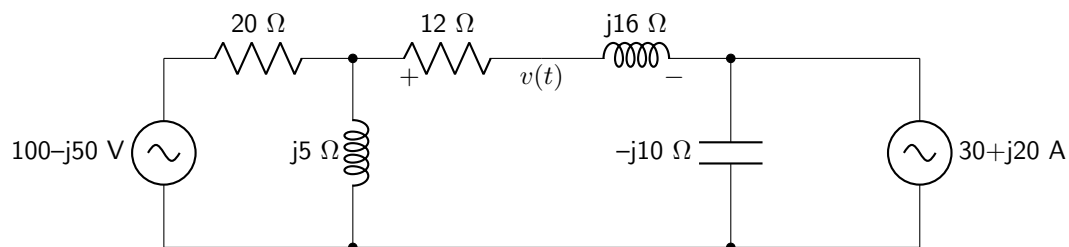


Figure 9.51: Circuit diagram for complex superposition question 18.

19. Calculate $i(t)$ given the circuit diagram shown in figure 9.52. The frequency of operation is 2 kHz.

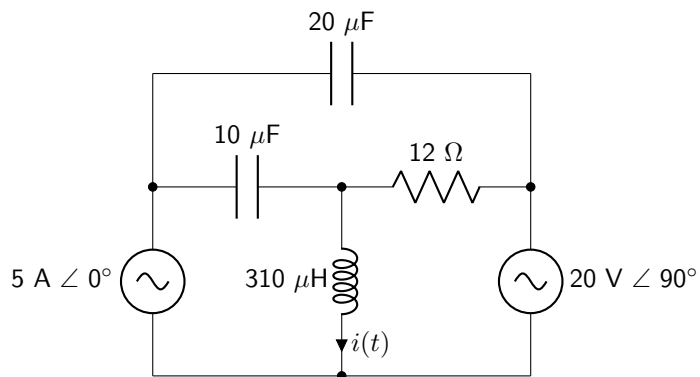


Figure 9.52: Circuit diagram for complex superposition question 19.

20. Calculate $i(t)$ given the circuit diagram shown in figure 9.53. The frequency of operation is 20 Hz.

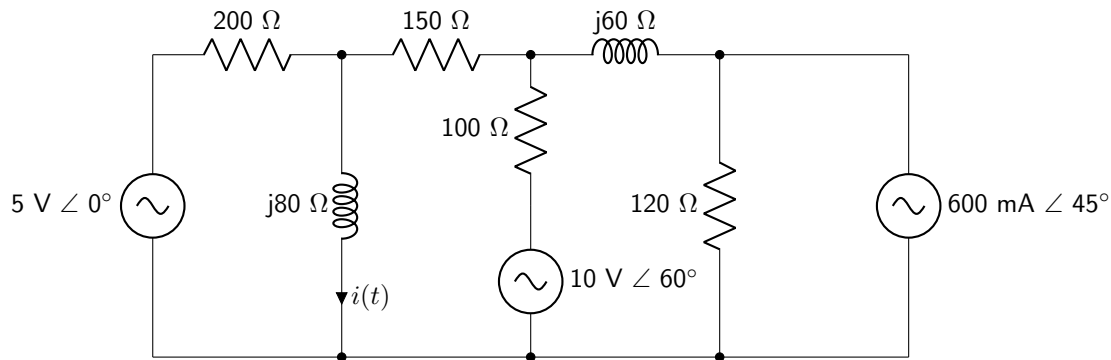


Figure 9.53: Circuit diagram for complex superposition question 20.

Complex Source Transformation

21. Use source transformation to calculate $v(t)$ in the circuit shown in figure 9.49 (in the complex superposition section).
22. Use source transformation to calculate $v(t)$ in the circuit shown in figure 9.50 (in the complex superposition section).
23. Use source transformation to calculate $i(t)$ for the circuit shown in figure 9.54. The frequency of operation is 10 Hz.

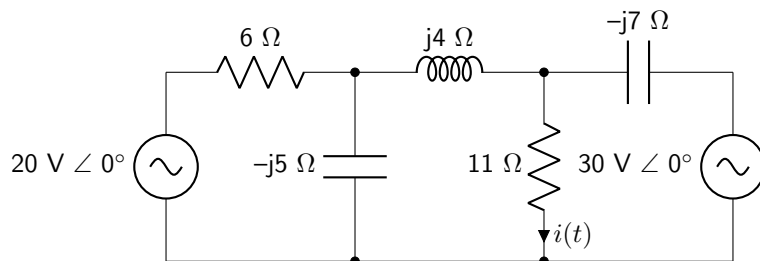


Figure 9.54: Circuit diagram for complex source transformation question 23.

24. Use source transformation to calculate $v(t)$ for the circuit shown in figure 9.55.

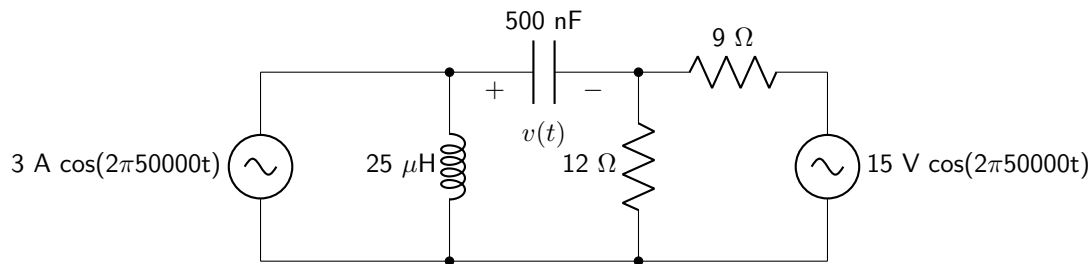


Figure 9.55: Circuit diagram for complex source transformation question 24.

25. Use source transformation to calculate $v(t)$ for the circuit shown in figure 9.56.

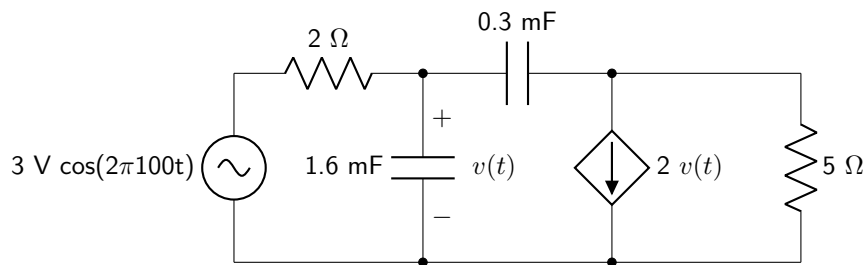


Figure 9.56: Circuit diagram for complex source transformation question 25.

Complex Thévenin and Norton's Theorems

26. Derive the Thévenin equivalent circuit between nodes a and b in the circuit shown in figure 9.57. The frequency of operation is 50 Hz.

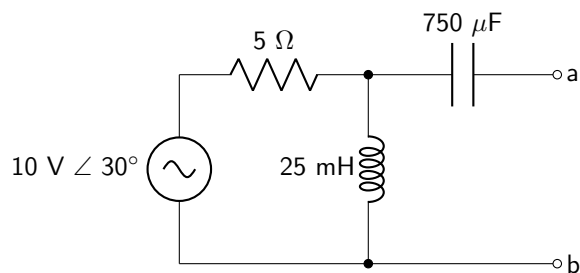


Figure 9.57: Circuit diagram for complex Thévenin and Norton theorems question 26.

27. Derive the Thévenin equivalent circuit between nodes a and b in the circuit shown in figure 9.58.

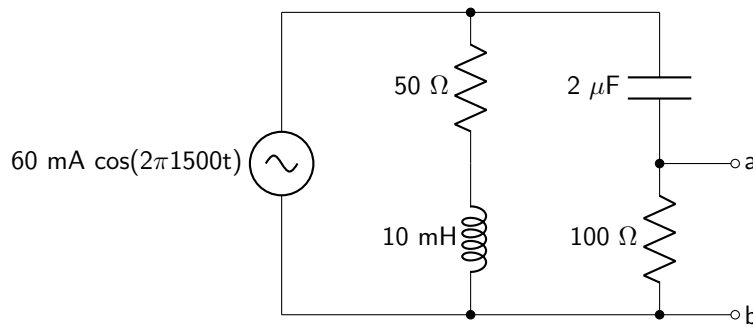


Figure 9.58: Circuit diagram for complex Thévenin and Norton theorems question 27.

28. Derive the Thévenin equivalent circuit between nodes a and b in the circuit shown in figure 9.59. The frequency of operation is 2 kHz.

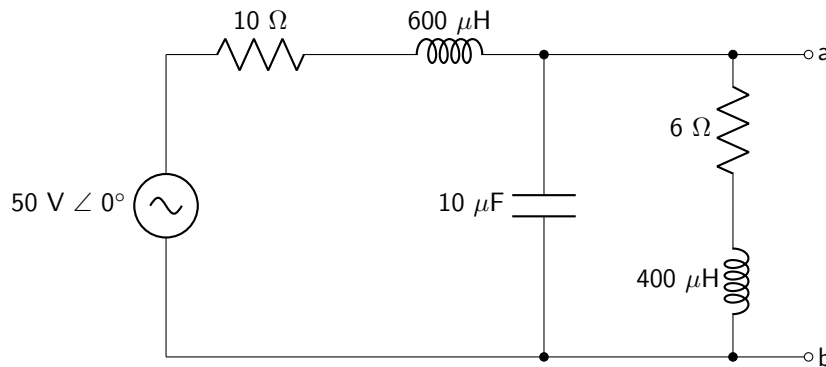


Figure 9.59: Circuit diagram for complex Thévenin and Norton theorems question 28.

29. Derive the Norton equivalent circuit between nodes a and b in the circuit shown in figure 9.60. The frequency of operation is 60 Hz.

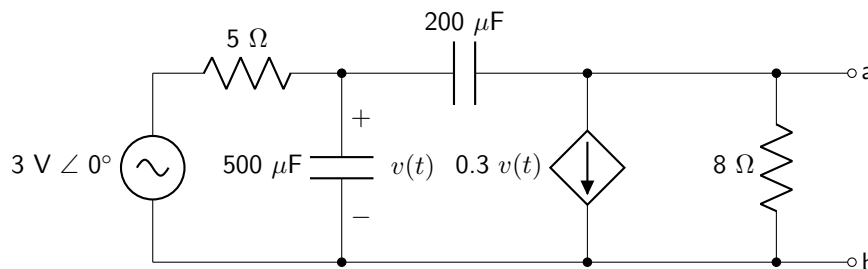


Figure 9.60: Circuit diagram for complex Thévenin and Norton theorems question 29.

30. Derive the Norton equivalent circuit between nodes a and b in the circuit shown in figure 9.61. The frequency of operation is 300 Hz.

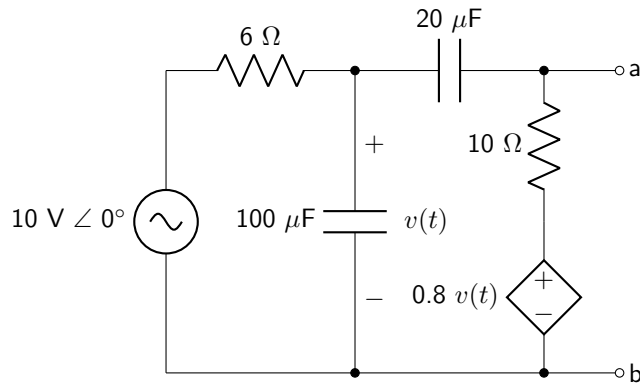


Figure 9.61: Circuit diagram for complex Thévenin and Norton theorems question 30.

Filters

31. Given the circuit shown in figure 9.62, determine the filter type, calculate the center frequency, bandwidth, and quality factor.

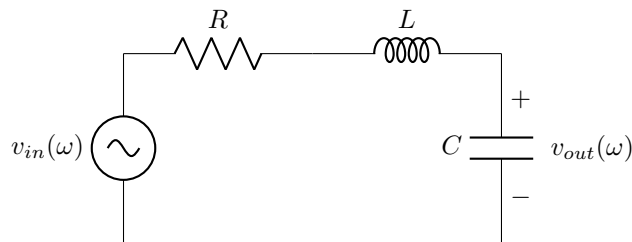


Figure 9.62: Circuit diagram for filters question 31.

32. Given the circuit shown in figure 9.63, determine the filter type, calculate the center frequency, bandwidth, and quality factor.

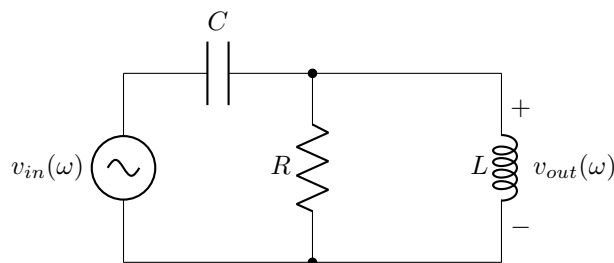


Figure 9.63: Circuit diagram for filters question 32.

33. Given the circuit shown in figure 9.64, determine the filter type, calculate the center frequency, bandwidth, and quality factor.

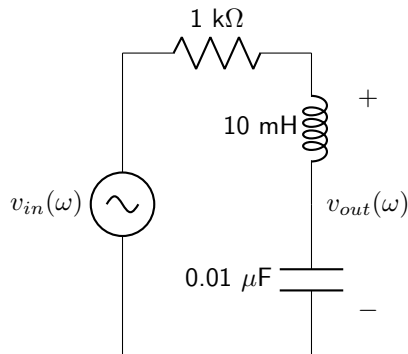


Figure 9.64: Circuit diagram for filters question 33.

34. Given the circuit shown in figure 9.65, determine the filter type and calculate the cutoff frequency.

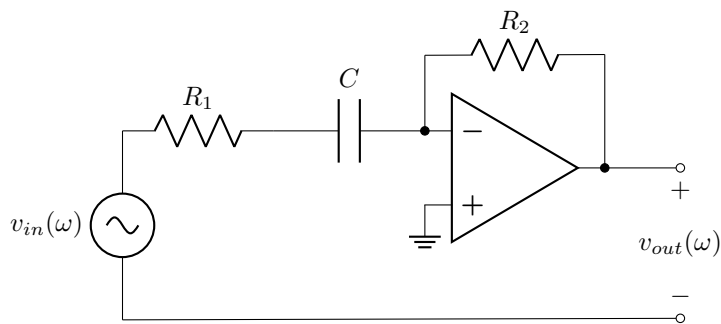


Figure 9.65: Circuit diagram for filters question 34.

35. Given the circuit shown in figure 9.66, determine the filter type and calculate the cutoff frequency.

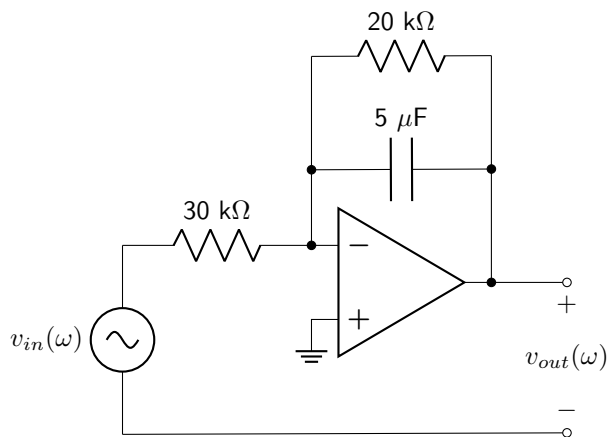


Figure 9.66: Circuit diagram for filters question 35.

Transfer Functions

36. Derive a transfer function for the circuit shown in figure 9.67. Then, determine the filter type.

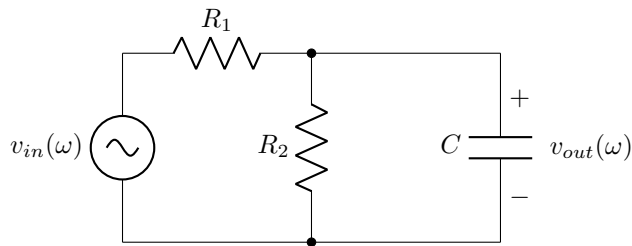


Figure 9.67: Circuit diagram for transfer functions question 36.

37. Derive a transfer function for the circuit shown in figure 9.68. Then, determine the filter type.

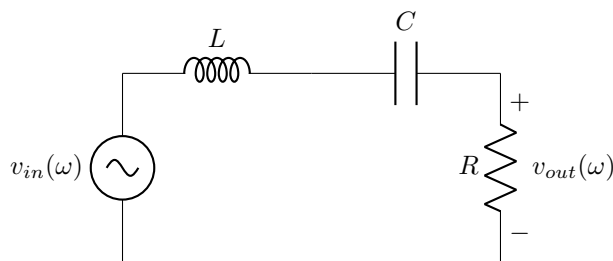


Figure 9.68: Circuit diagram for transfer functions question 37.

38. Derive a transfer function for the circuit shown in figure 9.69. Then, determine the filter type.

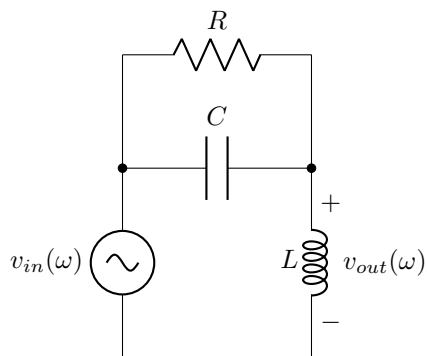


Figure 9.69: Circuit diagram for transfer functions question 38.

39. Derive a transfer function for the circuit shown in figure 9.70. Then, determine the filter type.

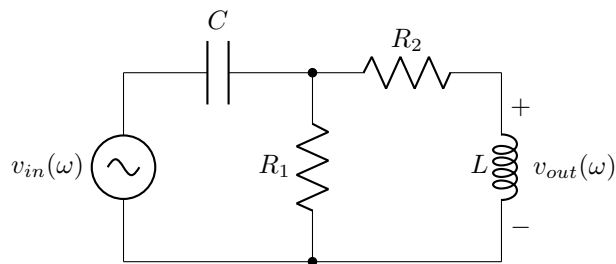


Figure 9.70: Circuit diagram for transfer functions question 39.

40. Derive a transfer function for the circuit shown in figure 9.71. Then, determine the filter type.

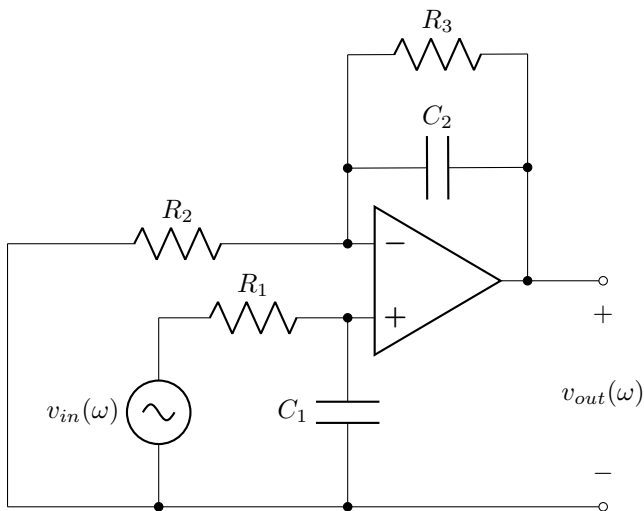


Figure 9.71: Circuit diagram for transfer functions question 40.

10 AC Power

10.1 Phase and Root Mean Square (RMS)

Discussions about power require an understanding of the voltage and current characteristics of a circuit. In DC circuits, this analysis is relatively straightforward because voltages and currents do not vary with respect to time. Even in transient analysis, changes occur quickly and return to a steady-state. This analysis becomes more difficult in AC circuits because elements do not settle down to one value; the sinusoidal nature of voltages and currents imply that values are continuously changing.

AC circuits may also have a *phase difference* between voltage and current signals. In other words: maximum current may arrive at the load slightly before maximum voltage, or maximum current may arrive at the load slightly after maximum voltage. This phase difference leads to *complex power*, which will be explained in the next section of this book. The phase difference is described in equation 10.1, where ϕ_v is the phase of the voltage and ϕ_i is the phase of the current.

$$\theta = \phi_v - \phi_i \quad (10.1)$$

Because time-varying signals are frequently plotted with respect to time, and not phase, the phase difference (in degrees) can be determined by calculating the time difference between subsequent peaks divided by the period of the wave (T), as defined in equation 10.2. To calculate the phase difference in radians, multiply by 2π instead of 360° .

$$\theta = 360^\circ \left(\frac{t_{peak,voltage} - t_{peak,current}}{T} \right) \quad (10.2)$$

Figure 10.1 shows two different voltage/current relationships measured in the load of a hypothetical circuit containing resistors, capacitors, and inductors.

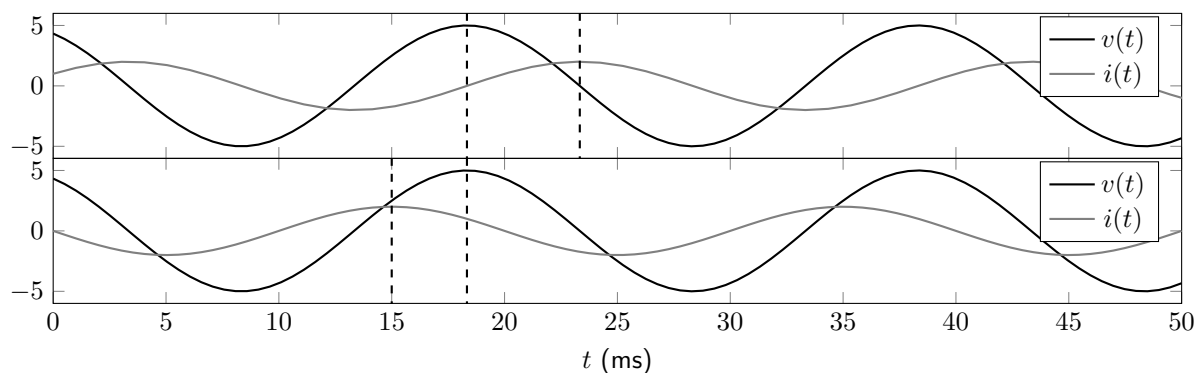


Figure 10.1: Current-voltage signals for a lagging (top) circuit and a leading (bottom) circuit.

In the top figure, the voltage arrives at the load before current. In other words: the current *lags* the

voltage. This is called a lagging circuit, and is caused by an inductive load. In the bottom figure, the voltage arrives at the load after current. In other words: the current *leads* the voltage. This is called a leading circuit, and is caused by a capacitive load.

So far, the amplitude of a wave (V_m for voltage and I_m for current) has been used to quantify sinusoidal signals. The amplitude defines the maximum displacement from equilibrium. The average value also provides useful information. However, a sinusoidal wave with no DC offset has an average of zero. This means that a normal average does not provide any useful information about a sine or cosine wave. The concept of *root mean square* (RMS) value is used instead.

The root mean square definition for a periodic waveform ($f(t)$) is defined in equation 10.3, where T is equal to the period of the function.

$$f_{RMS} = \sqrt{\frac{1}{T} \int_0^T f^2(t) dt} \quad (10.3)$$

The RMS value for a sinusoidal wave is therefore equal to the amplitude divided by the square root of two, shown in equation 10.4. (All of the voltage terms can be replaced with current terms to find the RMS value of a sinusoidal current.)

$$V_{RMS} = \frac{V_m}{\sqrt{2}} \quad (10.4)$$

Triangular and ramp waves have RMS values defined in equation 10.5.

$$V_{RMS} = \frac{V_m}{\sqrt{3}} \quad (10.5)$$

As will be seen in the next section, RMS values will be used in calculations of complex power in AC circuits. These values will be noted with a subscript on the variable names, and with the term RMS used in conjunction with units. In other words, $V_{RMS} = 10 \text{ VRMS}$ will be used to define an RMS voltage of 10 V.

10.2 AC Power

Power, which was initially described in section 1.4 of this textbook, is lost or gained at particular rates in each circuit element. In AC circuits, there are many different quantities used to describe the power that is delivered or absorbed by different circuit elements.

10.2.1 Instantaneous Power

Instantaneous power was defined in equation 1.2. Instantaneous power in a circuit can be calculated using the time-varying forms of voltage and current and multiplying them together. The instantaneous power equation, rewritten in terms of cosine form for voltage and current, is shown in equation 10.6 and derived

below. (Note, the derivation uses a product-to-sum trigonometric identity to derive the result.)

$$p(t) = V_{RMS}I_{RMS} [\cos \theta + \cos (2\omega t + \phi_v + \phi_i)] \quad (10.6)$$

DERIVATION

$$\begin{aligned} v(t) &= V_m \cos(\omega t + \phi_v) \\ i(t) &= I_m \cos(\omega t + \phi_i) \\ p(t) &= [V_m \cos(\omega t + \phi_v)][I_m \cos(\omega t + \phi_i)] \\ &= \frac{V_m I_m}{2} [\cos \theta + \cos(2\omega t + \phi_v + \phi_i)] \\ &= V_{RMS}I_{RMS} [\cos \theta + \cos(2\omega t + \phi_v + \phi_i)] \end{aligned}$$

10.2.2 Complex and Apparent Power

Complex power is defined in equation 10.7 and derived below, where the * symbol denotes the complex conjugate. The units used with complex power are VA (volt-amperes).

$$\mathbf{S} = \mathbf{V}_{RMS}\mathbf{I}_{RMS}^* \quad (10.7)$$

DERIVATION

$$\begin{aligned} \mathbf{S} &= V_{RMS}I_{RMS}e^{j\theta} \\ &= V_{RMS}I_{RMS}e^{j(\phi_v - \phi_i)} \\ &= V_{RMS}e^{j\phi_v}I_{RMS}e^{-j\phi_i} \\ &= \mathbf{V}_{RMS}\mathbf{I}_{RMS}^* \end{aligned}$$

Equation 10.7 can be unwieldy to use if only current is known and not voltage; or if only voltage is known and not current. If the voltage is known (but not the current), the equation can be rewritten in terms of only voltage and impedance. This method uses the complex version of Ohm's law and is shown in equation 10.8 and derived below.

$$\mathbf{S} = \frac{|\mathbf{V}_{RMS}|^2 \mathbf{Z}}{|\mathbf{Z}|^2} \quad (10.8)$$

DERIVATION

$$\begin{aligned}
 \mathbf{I}_{\text{RMS}}^* &= \left(\frac{\mathbf{V}_{\text{RMS}}}{\mathbf{Z}} \right)^* \\
 \mathbf{S} &= \frac{\mathbf{V}_{\text{RMS}} \mathbf{V}_{\text{RMS}}^*}{\mathbf{Z}^*} \\
 &= \frac{|\mathbf{V}_{\text{RMS}}|^2}{\mathbf{Z}^*} \\
 &= \frac{|\mathbf{V}_{\text{RMS}}|^2 \mathbf{Z}}{\mathbf{Z} \mathbf{Z}^*} \\
 &= \frac{|\mathbf{V}_{\text{RMS}}|^2 \mathbf{Z}}{|\mathbf{Z}|^2}
 \end{aligned}$$

If the current is known (but not the voltage), the equation can be rewritten as shown in equation 10.9 (and derived below) in terms of only current and impedance.

$$\mathbf{S} = |\mathbf{I}_{\text{RMS}}|^2 \mathbf{Z} \quad (10.9)$$

DERIVATION

$$\begin{aligned}
 \mathbf{V}_{\text{RMS}} &= \mathbf{I}_{\text{RMS}} \mathbf{Z} \\
 \mathbf{S} &= \mathbf{I}_{\text{RMS}} \mathbf{I}_{\text{RMS}}^* \mathbf{Z} \\
 &= |\mathbf{I}_{\text{RMS}}|^2 \mathbf{Z}
 \end{aligned}$$

In Cartesian form, the complex power is defined in equation 10.10. Both terms P (average power) and Q (reactive power) will be defined in the next two sections of this book.

$$\mathbf{S} = P + jQ \quad (10.10)$$

The magnitude of the complex power is known as *apparent power*. It is also measured in units of VA. It is defined in equation 10.11.

$$|\mathbf{S}| = V_{\text{RMS}} I_{\text{RMS}} = \sqrt{P^2 + Q^2} \quad (10.11)$$

10.2.3 Average Power

The average power is equal to the average of the instantaneous power (defined in equation 10.6) over one period. It is also equal to the real part of the complex power. Either way, the equation for average power is

provided in equation 10.12.

$$P = V_{RMS}I_{RMS} \cos \theta \quad (10.12)$$

Average power is the component of complex power that does useful work in a circuit. It is generally advantageous to maximize average power and reduce reactive power as much as possible.

10.2.4 Reactive Power

Reactive power can be conceptually understood as energy that oscillates between the load and any reactive (inductive or capacitive) components without doing any useful work in a circuit. It is equal to the imaginary part of the complex power. Reactive power, measured in VAR (volt-amps reactive), is defined in equation 10.13.

$$Q = V_{RMS}I_{RMS} \sin \theta \quad (10.13)$$

Because reactive power does not contribute any useful work to a circuit, it is generally advantageous to minimize it as much as possible.

10.2.5 Power Triangle and Power Factor

The power triangle is used to quantify the complex nature of power consumption. The real part of the power triangle, P , is the average power (measured in W). The imaginary part of the power triangle, Q , is the reactive power (measured in VAR). The sum of these two components is the complex power, \mathbf{S} (measured in VA). This power triangle is depicted in figure 10.2.

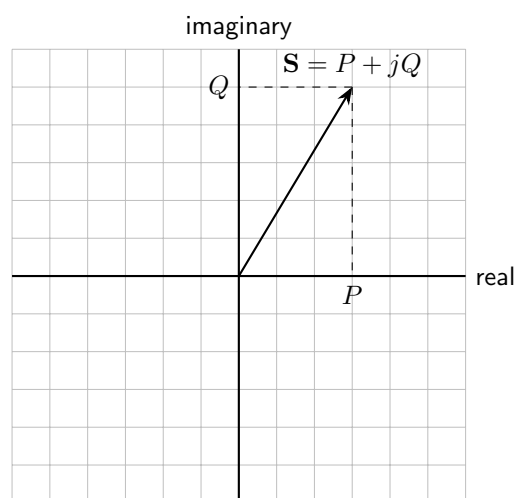


Figure 10.2: A power triangle is used to quantify the complex nature of AC power. The real part (P) is equal to the average power, while the imaginary part (Q) is equal to the reactive power.

Note that the units for each of the components is different (W for P , VAR for Q , VA for \mathbf{S}). The unit for power (regardless of the type of power) is watts. However, to clearly differentiate reactive power from average power from complex power, different units are used. VA (the units for complex power) stands for volt-amps. VAR (the units for reactive power) stands for volt-amps reactive. This means that you can look at a number and know exactly what type of power it corresponds to. 28 VAR would be a measure of complex power, whereas 28 W would be a measure of average power.

The angle of the complex power (measured in relation to the positive real axis, as described in section 8.1) can be used to quantify how much of the power is real vs. reactive. The quantity used to describe this is called the *power factor* and is defined in equation 10.14. Note that θ is the same angle that was calculated in equation 10.1.

$$pf = \cos \theta = \frac{P}{|\mathbf{S}|} \quad (10.14)$$

This power factor relates to how efficiently a circuit delivers useable power to a load. The closer the power factor is to one, the more the power is real. The closer the power factor is to zero, the more the power is reactive. Reactive power does no useful work in a circuit, but still uses energy (and thus costs money when connected to the power grid).

Example: Calculating complex power from time-varying voltage and current

Calculate the complex power consumed by a load characterized by the following voltage and current waveforms.

$$v(t) = 100 \text{ V} \cos(2\pi 60t + 120\pi/180)$$

$$i(t) = 8 \text{ A} \cos(2\pi 60t + 60\pi/180)$$

Calculate the phase offset between the voltage and current using equation 10.1.

$$\begin{aligned} \theta &= 120^\circ - 60^\circ \\ &= 60^\circ \end{aligned}$$

Calculate the RMS value of both the voltage and the current.

$$\begin{aligned} |\mathbf{V}| &= \frac{100}{\sqrt{2}} \text{ VRMS} \\ &= 70.71 \text{ VRMS} \\ |\mathbf{I}| &= \frac{8}{\sqrt{2}} \text{ ARMS} \\ &= 5.66 \text{ ARMS} \end{aligned}$$

Use equation 10.12 to calculate the real power.

$$\begin{aligned} P &= (70.71 \text{ VRMS})(5.66 \text{ ARMS}) \cos 60^\circ \\ &= 200 \text{ W} \end{aligned}$$

Use equation 10.13 to calculate the reactive power.

$$\begin{aligned} Q &= (70.71 \text{ VRMS})(5.66 \text{ ARMS}) \sin 60^\circ \\ &= 346.41 \text{ VAR} \end{aligned}$$

Use equation 10.10 to determine the complex power.

$$\mathbf{S} = 200 + j346.41 \text{ VA}$$

Example: Calculating complex power from average power and power factor

Calculate the complex power consumed by a load characterized by an average power consumption of 1630 W and a lagging power factor of 0.862.

Use equation 10.14 to calculate the apparent power.

$$\begin{aligned} |\mathbf{S}| &= \frac{1630 \text{ W}}{0.862} \\ &= 1890.95 \text{ VA} \end{aligned}$$

Use equation 10.10 to calculate the reactive power. It must be positive because the power factor is lagging.

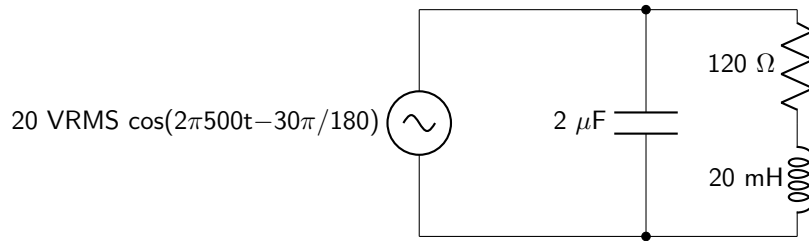
$$\begin{aligned} Q &= |\mathbf{S}| \sqrt{1 - pf^2} \\ &= (1890.95 \text{ VA}) \sqrt{1 - 0.862^2} \\ &= 958.54 \text{ VAR} \end{aligned}$$

It is now simple to state the complex power in terms of the average and reactive power.

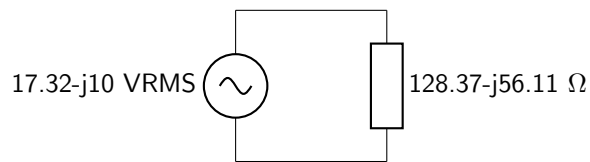
$$\mathbf{S} = 1890.95 + j958.54 \text{ VA}$$

Example: Calculating complex power from a circuit diagram

Calculate the complex power consumed by the following circuit.



First, phasor transform the circuit. The load impedance is equal to the impedance of the capacitor in parallel with the series combination of the resistor and inductor.



Use the complex form of Ohm's law to calculate the current flowing through the circuit.

$$\begin{aligned} \mathbf{I} &= \frac{\mathbf{V}}{\mathbf{Z}} \\ &= \frac{17.32 - j10 \text{ VRMS}}{128.37 - j56.11 \Omega} \\ &= 141.87 - j15.89 \text{ mARMS} \end{aligned}$$

Use equation 10.7 to calculate the complex power.

$$\begin{aligned} \mathbf{S} &= (17.32 - j10 \text{ VRMS})(141.87 + j15.89 \text{ mARMS}) \\ &= 2616.11 - j1143.48 \text{ mVA} \end{aligned}$$

10.3 Maximum Power Transfer

The theorem of maximum power transfer states that the maximum amount of power will be delivered to the load when the impedance of the load is equal to the complex conjugate of the Thévenin equivalent impedance of the circuit. The circuit shown in figure 10.3 will be used to demonstrate this theorem.

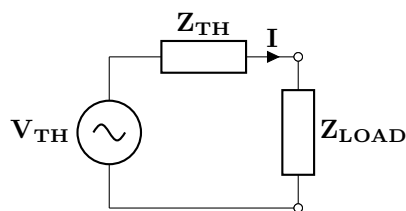


Figure 10.3: Circuit diagram used to demonstrate the theorem of maximum power transfer for AC circuits.

To determine the impedance of the load (\mathbf{Z}_{LOAD}) that will lead to maximum power transfer, the power

consumed by the load can be calculated using equation 10.7. This is shown in equation 10.15 and derived below.

$$\mathbf{S}_{\text{LOAD}} = \frac{|\mathbf{V}_{\text{TH,RMS}}|^2 \mathbf{Z}_{\text{LOAD}}}{|\mathbf{Z}_{\text{TH}} + \mathbf{Z}_{\text{LOAD}}|^2} \quad (10.15)$$

DERIVATION

$$\begin{aligned} \mathbf{I}_{\text{LOAD,RMS}} &= \frac{\mathbf{V}_{\text{TH,RMS}}}{\mathbf{Z}_{\text{TH}} + \mathbf{Z}_{\text{LOAD}}} \\ \mathbf{V}_{\text{LOAD,RMS}} &= \mathbf{V}_{\text{TH,RMS}} \left(\frac{\mathbf{Z}_{\text{LOAD}}}{\mathbf{Z}_{\text{TH}} + \mathbf{Z}_{\text{LOAD}}} \right) \\ \mathbf{S}_{\text{LOAD}} &= \mathbf{V}_{\text{LOAD,RMS}} \mathbf{I}_{\text{LOAD,RMS}}^* \\ &= \mathbf{V}_{\text{TH,RMS}} \left(\frac{\mathbf{Z}_{\text{LOAD}}}{\mathbf{Z}_{\text{TH}} + \mathbf{Z}_{\text{LOAD}}} \right) \left(\frac{\mathbf{V}_{\text{TH,RMS}}^*}{(\mathbf{Z}_{\text{TH}} + \mathbf{Z}_{\text{LOAD}})^*} \right) \\ &= \frac{|\mathbf{V}_{\text{TH,RMS}}|^2 \mathbf{Z}_{\text{LOAD}}}{|\mathbf{Z}_{\text{TH}} + \mathbf{Z}_{\text{LOAD}}|^2} \end{aligned}$$

The partial derivatives of equation 10.15 can be taken with respect to R_{LOAD} and X_{LOAD} and set equal to zero to find the maximum. (These derivations will not be shown in this book.) The maximum occurs when R_{LOAD} is equal to R_{TH} , and X_{LOAD} is equal to $-X_{\text{TH}}$. In other words, \mathbf{Z}_{LOAD} is equal to \mathbf{Z}_{TH}^* .

The maximum power that can be consumed by the load can then be calculated by solving equation 10.15 for $\mathbf{Z}_{\text{LOAD}} = R_{\text{TH}} - jX_{\text{TH}}$. This is shown in equation 10.16 and derived below.

$$P_{\text{LOAD,MAX}} = \frac{|\mathbf{V}_{\text{TH,RMS}}|^2}{4R_{\text{TH}}} \quad (10.16)$$

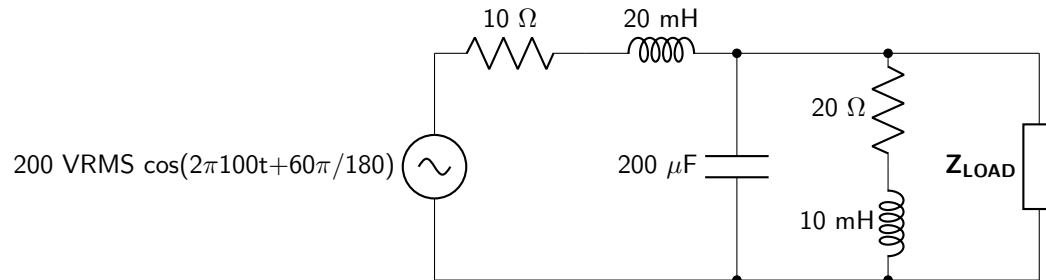
DERIVATION

$$\begin{aligned} \mathbf{S}_{\text{LOAD,MAX}} &= \frac{|\mathbf{V}_{\text{TH,RMS}}|^2 (R_{\text{TH}} - jX_{\text{TH}})}{(2R_{\text{TH}})^2} \\ &= \frac{|\mathbf{V}_{\text{TH,RMS}}|^2 R_{\text{TH}}}{(2R_{\text{TH}})^2} - j \frac{|\mathbf{V}_{\text{TH,RMS}}|^2 X_{\text{TH}}}{(2R_{\text{TH}})^2} \\ &= \frac{|\mathbf{V}_{\text{TH,RMS}}|^2}{4R_{\text{TH}}} - j \frac{|\mathbf{V}_{\text{TH,RMS}}|^2 X_{\text{TH}}}{4R_{\text{TH}}^2} \\ P_{\text{LOAD,MAX}} &= \frac{|\mathbf{V}_{\text{TH,RMS}}|^2}{4R_{\text{TH}}} \end{aligned}$$

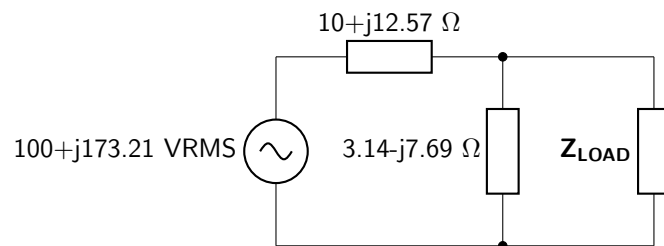
Because a Norton equivalent circuit and a Thévenin equivalent circuit are simply source-transformed versions of each other, the maximum power transferred to the load in a Norton equivalent circuit would occur when the load impedance is equal to the complex conjugate of the Norton impedance.

Example: Deriving a load for maximum power transfer

Determine the components of the load, and their corresponding values, that will lead to maximum power transfer.



First, phasor transform the circuit.



Deactivate the source to calculate the Thévenin equivalent impedance. This will be equal to both impedances in parallel.

$$\begin{aligned} \mathbf{Z}_{\text{TH}} &= \frac{(10 + j12.57 \Omega)(3.14 - j7.69 \Omega)}{(10 + j12.57 \Omega) + (3.14 - j7.69 \Omega)} \\ &= 7.64 - j5.68 \Omega \end{aligned}$$

The resistance of the optimal load is equal to the real part of \mathbf{Z}_{TH} . The reactance of the optimal load is equal to the negative value of the imaginary part of \mathbf{Z}_{TH} . This requires the use of an inductor. The value of the inductor can be calculated using equation 8.18.

$$\begin{aligned} L &= \frac{5.68 \Omega}{2\pi 100 \text{ Hz}} \\ &= 9 \text{ mH} \end{aligned}$$

The load that will lead to maximum power transfer is therefore a resistance of 7.64Ω in series with a 9 mH inductor.

10.4 Power Factor Correction

It is possible to minimize the reactive power present in the load of a circuit. This is accomplished by putting a compensation circuit in parallel with the load, as depicted in figure 10.4.

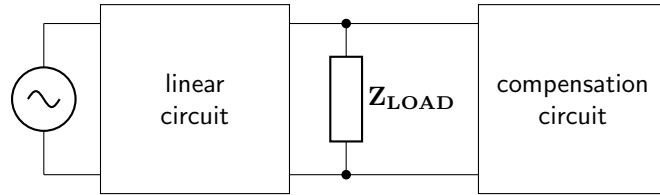


Figure 10.4: A compensation circuit can be used to increase the power factor by reducing complex power consumed by the load.

10.4.1 Correcting an Inductive Load

To correct an inductive load, a capacitor can be placed in parallel with the load. The value of capacitance that is required depends on the final power factor value that is desired (pf_f). The final reactive power (Q_f) that is necessary to achieve this final power factor can then be calculated by solving equations 10.14 and 10.11 for Q . This result will not be derived in this book but is shown in equation 10.17. (Note that the sign of Q_f is positive because the load will still be inductive after correction, although only slightly. If pf_f is one, then the load will be resistive and Q_f will be equal to zero.)

$$Q_f = P \sqrt{\frac{1}{pf_f^2} - 1} \quad (10.17)$$

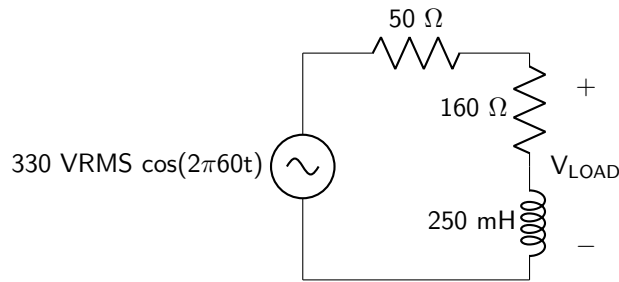
The compensation circuit therefore requires a reactive power that is capable of changing the reactive power from the initial value to the final value given in equation 10.17. That is, $Q_C = \Delta Q = Q_f - Q_i$. To find the value of capacitor that can generate Q_C , solve equation 10.8 for $Z = -\frac{j}{\omega C}$ and solve for C . This result is shown in equation 10.18.

$$C = \frac{-Q_C}{|\mathbf{V}_{\text{LOAD,RMS}}|^2 \omega} \quad (10.18)$$

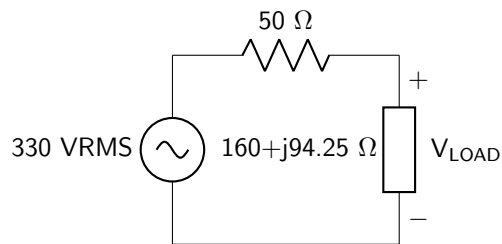
Inductive loads are very common in any circuitry that uses motors. Compensation circuits are frequently used to reduce the reactive power generated by the inductive load and reduce energy bills.

Example: Correcting an inductive load

Find a capacitor to place in parallel with the load to increase the power factor to 0.98.



Phasor transform the circuit.



Use the voltage divider tool to calculate the voltage drop over the load. (This is required to determine the original reactive power.) It is also useful to calculate the magnitude of the load voltage at this time.

$$\begin{aligned} \mathbf{V}_{\text{LOAD,RMS}} &= 330 \text{ VRMS} \left(\frac{160 + j94.25 \Omega}{210 + j94.25 \Omega} \right) \\ &= 264.60 + j29.35 \text{ VRMS} \\ |\mathbf{V}_{\text{LOAD,RMS}}|^2 &= 70875.28 \text{ VRMS}^2 \end{aligned}$$

Use equation 10.8 to calculate the power consumed by the load.

$$\begin{aligned} \mathbf{S}_{\text{LOAD}} &= \frac{(70875.28 \text{ VRMS}^2)(160 + j94.25 \Omega)}{|(160 + j94.25 \Omega)|^2} \\ &= \frac{(70875.28 \text{ VRMS}^2)(160 + j94.25 \Omega)}{34482.64 \Omega^2} \\ &= 328.86 + j193.72 \text{ VA} \end{aligned}$$

The initial reactive power is 193.72 VAR. Use equation 10.17 to calculate the final reactive power.

$$\begin{aligned} Q_f &= (328.86 \text{ W}) \sqrt{\frac{1}{0.98^2} - 1} \\ &= 66.78 \text{ VAR} \end{aligned}$$

The change in reactive power due to the compensating capacitor must be 66.78 VAR – 193.72 VAR =

−126.94 VAR. Use equation 10.18 to determine the value of the compensating capacitor.

$$\begin{aligned} C &= \frac{126.94 \text{ VAR}}{(70875.28 \text{ VRMS}^2)(2\pi 60 \text{ Hz})} \\ &= 4.75 \times 10^{-6} \text{ F} \\ &= 4.75 \mu\text{F} \end{aligned}$$

10.4.2 Correcting a Capacitive Load

To correct a capacitive load, an inductor can be placed in parallel with the load. The value of inductance that is required depends on the final power factor value that is desired (pf_f). The final reactive power (Q_f) that is necessary to achieve this final power factor can then be calculated by solving equations 10.14 and 10.11 for Q . This is equal to the negative value of equation 10.17, shown in equation 10.19. (The sign now is negative because, after correction, the load will either be purely resistive or be slightly capacitive.)

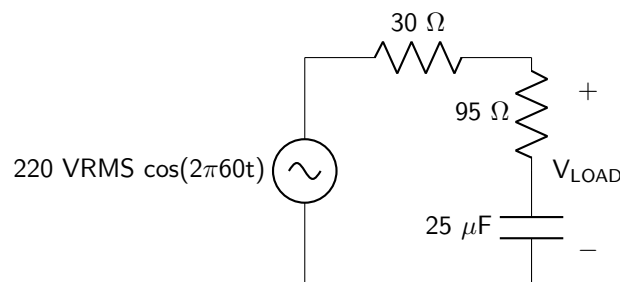
$$Q_f = -P \sqrt{\frac{1}{pf_f^2} - 1} \quad (10.19)$$

The compensation circuit therefore requires a reactive power that is capable of changing the reactive power from the initial value to the final value given in equation 10.19. That is, $Q_L = \Delta Q = Q_f - Q_i$. To find the value of inductor that can generate Q_L , solve equation 10.8 for $Z = j\omega L$ and solve for L . This result is shown in equation 10.20.

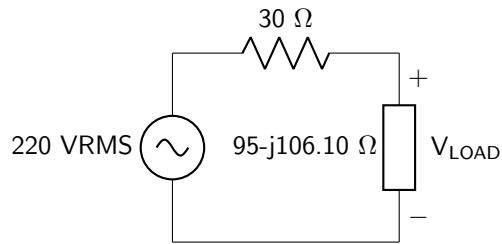
$$L = \frac{|V_{\text{LOAD,RMS}}|^2}{\omega Q_L} \quad (10.20)$$

Example: Correcting a capacitive load

Find an inductor to place in parallel with the load to increase the power factor to 0.99.



Phasor transform the circuit.



Use the voltage divider tool to calculate the voltage drop over the load. (This is required to determine the original reactive power.) It is also useful to calculate the magnitude of the load voltage at this time.

$$\begin{aligned} V_{\text{LOAD,RMS}} &= 220 \text{ VRMS} \left(\frac{95 - j106.10 \Omega}{125 - j106.10 \Omega} \right) \\ &= 189.31 - j26.05 \text{ VRMS} \\ |V_{\text{LOAD,RMS}}|^2 &= 36517.36 \text{ VRMS}^2 \end{aligned}$$

Use equation 10.8 to calculate the power consumed by the load.

$$\begin{aligned} S_{\text{LOAD}} &= \frac{(36517.36 \text{ VRMS}^2)(95 - j106.10 \Omega)}{|(95 - j106.10 \Omega)|^2} \\ &= \frac{(36517.36 \text{ VRMS}^2)(95 - j106.10 \Omega)}{20282.91 \Omega^2} \\ &= 171.04 - j191.03 \text{ VA} \end{aligned}$$

The initial reactive power is -191.03 VAR . Use equation 10.19 to calculate the final reactive power.

$$\begin{aligned} Q_f &= -(171.04 \text{ W}) \sqrt{\frac{1}{0.99^2} - 1} \\ &= -24.37 \text{ VAR} \end{aligned}$$

The change in reactive power due to the compensating inductor must be $-24.37 \text{ VAR} + 191.03 \text{ VAR} = 166.66 \text{ VAR}$. Use equation 10.20 to determine the value of the compensating inductor.

$$\begin{aligned} L &= \frac{(36517.36 \text{ VRMS}^2)}{(166.66 \text{ VAR})(2\pi 60 \text{ Hz})} \\ &= 0.581 \text{ H} \\ &= 581.23 \text{ mH} \end{aligned}$$

Example Problems

Phase and Root Mean Square

1. Calculate the RMS value of the voltage and current signals, as well as the phase difference. $v(t) = 120 \text{ V} \cos(2\pi 60t + 120\pi/180)$, $i(t) = 5 \text{ A} \cos(2\pi 60t + 60\pi/180)$
2. Calculate the RMS value of the voltage and current signals, as well as the phase difference. $v(t) = 150 \text{ V} \cos(2\pi 60t + 60\pi/180)$, $i(t) = 3 \text{ A} \cos(2\pi 60t + 35\pi/180)$
3. Calculate the RMS value of the voltage and current signals, as well as the phase difference. $v(t) = 220 \text{ V} \cos(2\pi 50t + 120\pi/180)$, $i(t) = 6 \text{ A} \cos(2\pi 50t - 40\pi/180)$
4. Calculate the RMS value of the voltage and current signals, as well as the phase difference, given the circuit shown in figure 10.5.

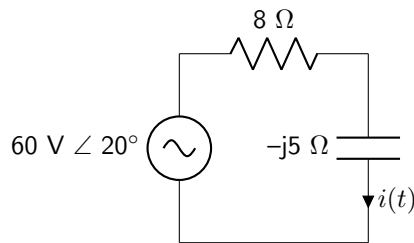


Figure 10.5: Circuit diagram for phase and root mean square question 4.

5. Calculate the RMS value of the voltage and current signals, as well as the phase difference, given the circuit shown in figure 10.6.

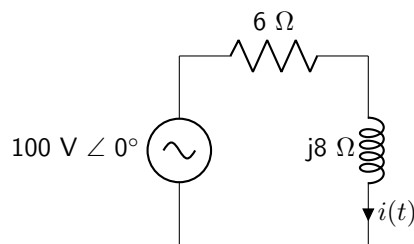


Figure 10.6: Circuit diagram for phase and root mean square question 5.

Complex Power

6. Calculate the power consumed by the load (indicated with dashed lines), as well as the power factor, for the circuit given in figure 10.7. The frequency of operation is 50 Hz.

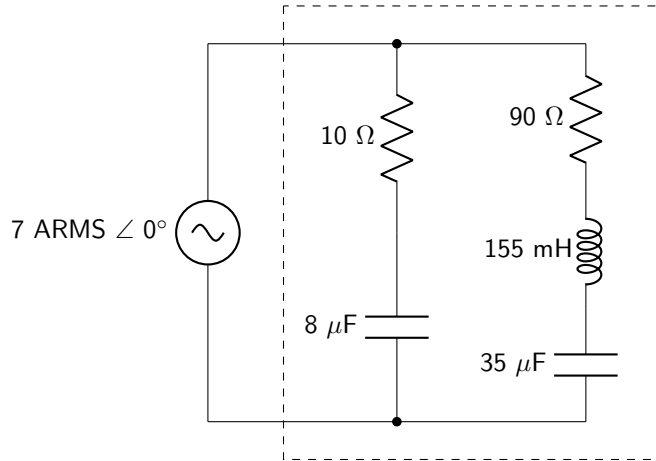


Figure 10.7: Circuit diagram for complex power question 6.

7. Calculate the power consumed by the load (indicated with dashed lines), as well as the power factor, for the circuit given in figure 10.8. The frequency of operation is 60 Hz.

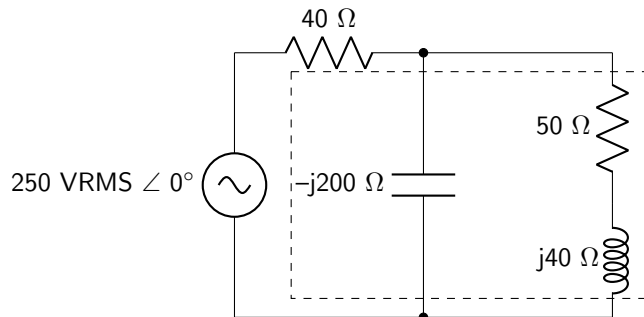


Figure 10.8: Circuit diagram for complex power question 7.

8. Calculate the power consumed by the load (indicated with dashed lines), as well as the power factor, for the circuit given in figure 10.9. The frequency of operation is 60 Hz.

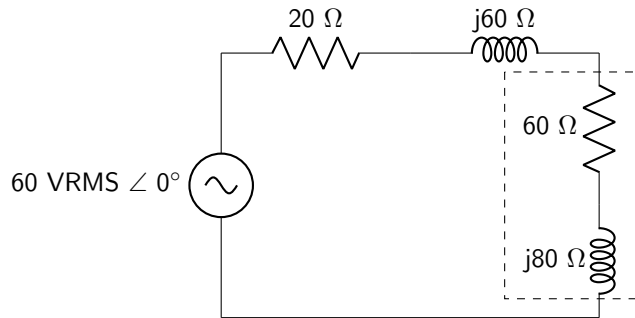


Figure 10.9: Circuit diagram for complex power question 8.

9. Calculate the power consumed by the load (indicated with dashed lines), as well as the power factor, for the circuit given in figure 10.10. The frequency of operation is 60 Hz.

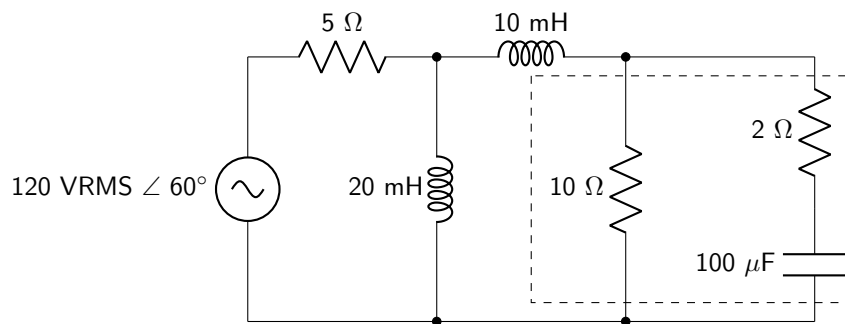


Figure 10.10: Circuit diagram for complex power question 9.

10. Calculate the power consumed by the load (indicated with dashed lines), as well as the power factor, for the circuit given in figure 10.11. The frequency of operation is 50 Hz.

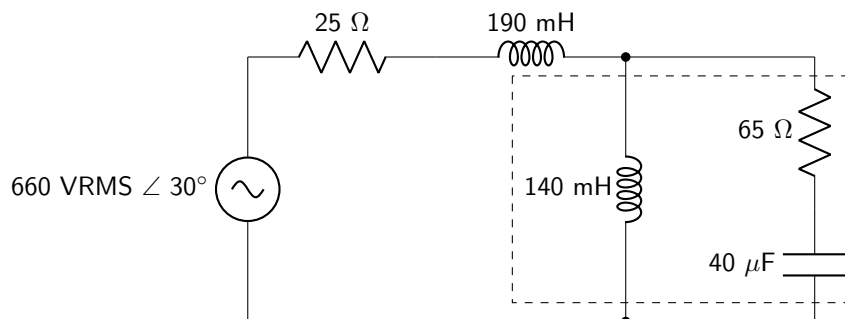


Figure 10.11: Circuit diagram for complex power question 10.

Maximum Power Transfer

11. Determine the circuit elements that must be placed on the load (shown as a generic circuit element) for maximum power transfer given the circuit in figure 10.12. Then, calculate the power consumed by that load. The frequency of operation is 60 Hz.

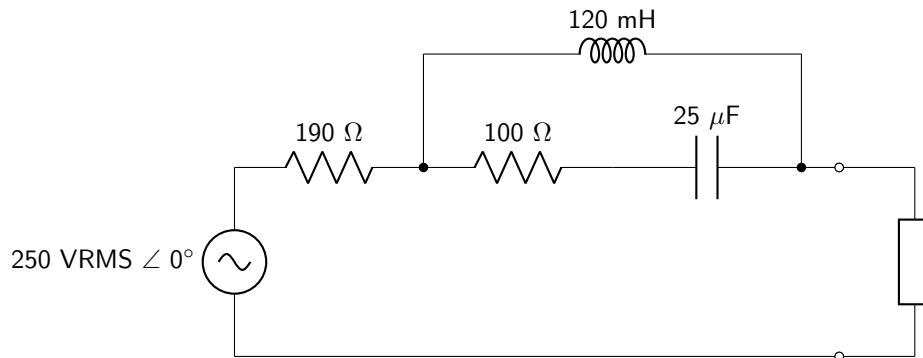


Figure 10.12: Circuit diagram for maximum power transfer question 11.

12. Determine the circuit elements that must be placed on the load (shown as a generic circuit element) for maximum power transfer given the circuit in figure 10.13. Then, calculate the power consumed by that load. The frequency of operation is 60 Hz.

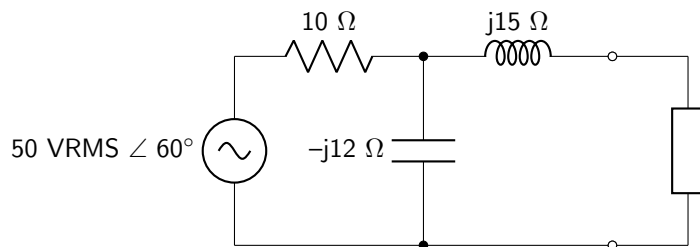


Figure 10.13: Circuit diagram for maximum power transfer question 12.

13. Determine the circuit elements that must be placed on the load (shown as a generic circuit element) for maximum power transfer given the circuit in figure 10.14. Then, calculate the power consumed by that load. The frequency of operation is 50 Hz.

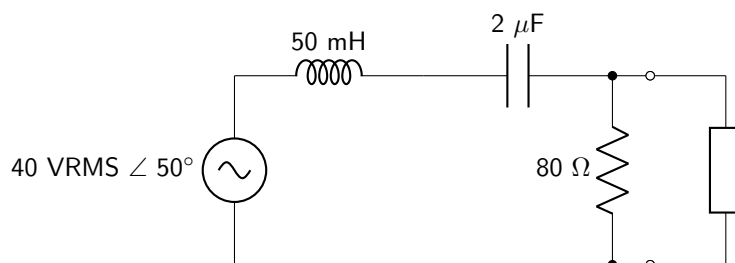


Figure 10.14: Circuit diagram for maximum power transfer question 13.

14. Determine the circuit elements that must be placed on the load (shown as a generic circuit element) for maximum power transfer given the circuit in figure 10.15. Then, calculate the power consumed by that load. The frequency of operation is 50 Hz.

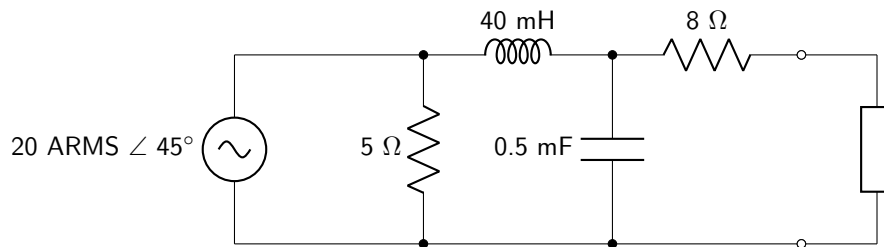


Figure 10.15: Circuit diagram for maximum power transfer question 14.

15. Determine the circuit elements that must be placed on the load (shown as a generic circuit element) for maximum power transfer given the circuit in figure 10.16. Then, calculate the power consumed by that load. The frequency of operation is 60 Hz.

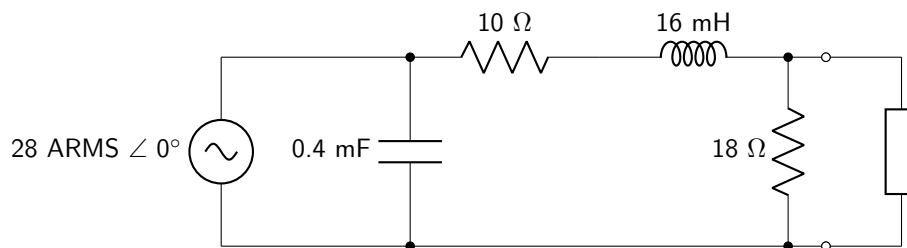


Figure 10.16: Circuit diagram for maximum power transfer question 15.

Power Factor Correction

16. Calculate the initial power factor consumed by the load (indicated with dashed lines) in the circuit shown in figure 10.17. Then, determine the circuit element that must be placed in parallel with the load to increase the power factor to 0.800. The frequency of operation is 60 Hz.

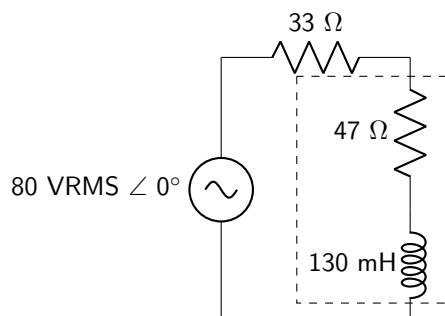


Figure 10.17: Circuit diagram for power factor correction question 16.

17. Calculate the initial power factor consumed by the load (indicated with dashed lines) in the circuit shown in figure 10.18. Then, determine the circuit element that must be placed in parallel with the load to increase the power factor to 0.860. The frequency of operation is 60 Hz.

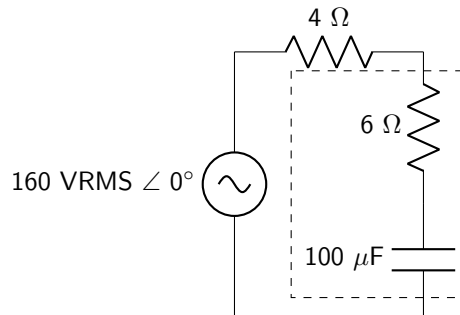


Figure 10.18: Circuit diagram for power factor correction question 17.

18. Calculate the initial power factor consumed by the load (indicated with dashed lines) in the circuit shown in figure 10.19. Then, determine the circuit element that must be placed in parallel with the load to increase the power factor to 0.990. The frequency of operation is 60 Hz.

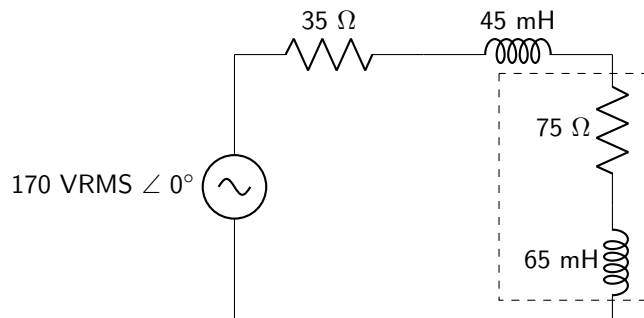


Figure 10.19: Circuit diagram for power factor correction question 18.

19. Calculate the initial power factor consumed by the load (indicated with dashed lines) in the circuit shown in figure 10.20. Then, determine the circuit element that must be placed in parallel with the load to increase the power factor to 0.980. The frequency of operation is 60 Hz.

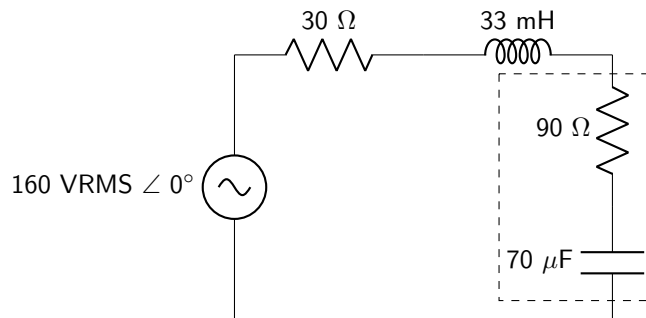


Figure 10.20: Circuit diagram for power factor correction question 19.

20. Calculate the initial power factor consumed by the load (indicated with dashed lines) in the circuit shown in figure 10.21. Then, determine the circuit element that must be placed in parallel with the load to increase the power factor to 0.960. The frequency of operation is 60 Hz.

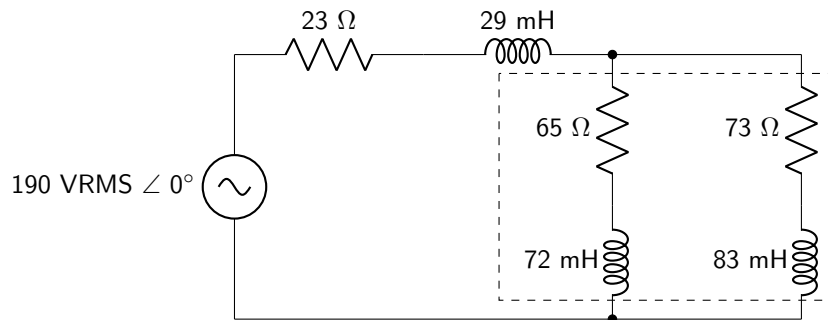


Figure 10.21: Circuit diagram for power factor correction question 20.

11 Laplace Transforms and s-Domain Analysis

11.1 The Laplace Transform

The Laplace transform, defined in equation 11.1, is used to simplify the analysis of circuits containing energy-storage elements such as capacitors and inductors. As was the case in phasor analysis, using the Laplace transform converts calculus to algebra. The inverse Laplace transform additionally aids in deriving time-domain equations for voltage and current without having to derive differential equations.

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st} dt \quad (11.1)$$

The variable s is a complex variable defined by equation 11.2. As discussed in section 9.4.2 of this book (although we did not formally use the variable s yet at that point), s relates to the location of a pole or zero on the complex axis. The real part (σ) relates to decaying exponential terms; the imaginary part (ω) relates to oscillatory (sinusoidal) terms. Performing a Laplace transform converts a time-domain function into the s-domain. (Phasor analysis essentially performed an equivalent transformation, where $\sigma = 0$.)

$$s = \sigma + j\omega \quad (11.2)$$

While Laplace transforms are especially important in many applications (such as signal processing), their focus will be limited to analysis of linear circuits in this textbook. As a result, some Laplace transform properties (convolution, for example) will not be covered in this textbook.

This textbook will not rigorously derive Laplace transforms for various elementary functions. Instead, Laplace transforms used for several frequently-used time-varying functions are given in table 11.1. These transforms will be directly applied to functions in the time domain to convert them to the s-domain, without requiring the direct use of equation 11.1 (which frequently necessitates the use of integration by parts). The reader is encouraged to derive the equations themselves if they would like to convince themselves of their veracity.

Function	Laplace Transform
$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$t u(t)$	$\frac{1}{s^2}$
$e^{-\alpha t} u(t)$	$\frac{1}{s+\alpha}$
$t e^{-\alpha t} u(t)$	$\frac{1}{(s+\alpha)^2}$
$\sin(\omega t) u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t) u(t)$	$\frac{s}{s^2+\omega^2}$
$e^{-\alpha t} \sin(\omega t) u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t} \cos(\omega t) u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Table 11.1: Laplace transforms of commonly-used time-varying functions.

Example: Laplace transform

Derive the Laplace transform of

$$f(t) = e^{-10t} u(t).$$

The Laplace transform is

$$F(s) = \frac{1}{s+10}.$$

The Laplace transform has many properties that can also be used to transform more complicated functions into the s-domain. These properties include multiplication, addition, and differentiation, and are included in table 11.2.

Property	Laplace Transform
Multiplication	$\mathcal{L}\{Kf(t)\} = KF(s)$
Addition and subtraction	$\mathcal{L}\{f_1(t) \pm f_2(t)\} = F_1(s) \pm F_2(s)$
Derivative	$\mathcal{L}\{\frac{d}{dt}f(t)\} = sF(s) - f(0^-)$
Second derivative	$\mathcal{L}\{\frac{d^2}{dt^2}f(t)\} = s^2F(s) - sf(0^-) - f'(0)$
Integral	$\mathcal{L}\{\int_0^t f(x) dx\} = \frac{F(s)}{s}$
Time translation	$\mathcal{L}\{f(t-\alpha) u(t-\alpha)\} = e^{-\alpha s} F(s), \alpha > 0$
Frequency translation	$\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s+\alpha)$
Scaling	$\mathcal{L}\{f(\alpha t)\} = \frac{1}{\alpha} F(\frac{s}{\alpha})$

Table 11.2: Properties of the Laplace transform.

Using tables 11.1 and 11.2, Laplace transforms of every time-domain function derived in this book can be computed.

Example: Laplace transform

Derive the Laplace transform of the voltage dropped over a circuit element. The time-domain response has been derived as

$$v(t) = [40 \text{ V} + e^{-10t} (2 \text{ V} \cos(20t) + 1 \text{ V} \sin(20t))] u(t).$$

Use the addition and subtraction property to analyze each individual term. (Note that units are being dropped from this point forward.)

$$V(s) = \mathcal{L}\{40 u(t)\} + \mathcal{L}\{2e^{-10t} \cos(20t) u(t)\} + \mathcal{L}\{e^{-10t} \sin(20t) u(t)\}$$

Use the multiplication property to factor out coefficients.

$$V(s) = 40\mathcal{L}\{u(t)\} + 2\mathcal{L}\{e^{-10t} \cos(20t) u(t)\} + \mathcal{L}\{e^{-10t} \sin(20t) u(t)\}$$

Use table 11.1 to calculate the Laplace transform.

$$V(s) = 40\frac{1}{s} + 2\frac{(s+10)}{(s+10)^2 + 20^2} + \frac{20}{(s+10)^2 + 20^2}$$

Find a common denominator for all terms and multiply through.

$$V(s) = \frac{40[(s+10)^2 + 400] + 2(s+10)s + 20s}{s[(s+10)^2 + 400]}$$

Simplify so there is a simple polynomial in both the numerator and the denominator. Factor the denominator so that the order of the polynomial is two.

$$V(s) = \frac{42s^2 + 840s + 20000}{s(s^2 + 20s + 500)}$$

Example: Laplace transform of a differential equation

Derive the Laplace transform of the current described by the differential equation in the example on page 195.

The differential equation is

$$0 = \frac{d^2i(t)}{dt^2} + 250\frac{di(t)}{dt} + 10500i(t).$$

The initial current flow $i(0) = 0.1 \text{ A}$ and the first derivative of the current flow is zero.

Use the addition and subtraction property to analyze each individual term.

$$0 = \mathcal{L}\left\{\frac{d^2i(t)}{dt^2}\right\} + \mathcal{L}\left\{250\frac{di(t)}{dt}\right\} + \mathcal{L}\{10500i(t)\}$$

Use the multiplication property to factor out coefficients.

$$0 = \mathcal{L}\left\{\frac{d^2i(t)}{dt^2}\right\} + 250\mathcal{L}\left\{\frac{di(t)}{dt}\right\} + 10500\mathcal{L}\{i(t)\}$$

Use the derivative and second derivative properties to find an expression in terms of $I(s)$.

$$0 = s^2I(s) - si(0) - i'(0) + 250[sI(s) - i(0)] + 10500I(s)$$

Plug in the initial conditions and simplify to find an expression for $I(s)$.

$$\begin{aligned} 0 &= s^2I(s) - 0.1s + 250[sI(s) - 0.1] + 10500I(s) \\ 0.1s + 25 &= s^2I(s) + 250sI(s) + 10500I(s) \\ I(s) &= \frac{0.1s + 25}{s^2 + 250s + 10500} \end{aligned}$$

11.2 Inverse Laplace Transform

To most easily derive a time-domain function of current or voltage in a circuit, an analysis can be done in the s-domain (which will be described in the next section) to derive an s-domain function $V(s)$ or $I(s)$. That s-domain function can then be inverse transformed back into the time domain. That inverse transform process will be described in this section.

Rather than applying a mathematical expression (such as that defined in equation 11.1), a Laplace transform is broken into individual components using partial fraction expansion. Then, table 11.1 can be used to derive the time-domain expression.

11.2.1 Partial Fraction Expansion

A function in the s-domain can be represented as a polynomial in the numerator and a polynomial in the denominator. The denominator can then be factored. Similar to equation 9.27, the factored denominator contains the poles of the circuit. A generalized factored s-domain expression is given in equation 11.3, where $N(s)$ is the numerator and p are the poles of the denominator.

$$F(s) = \frac{N(s)}{(s - p_1)(s - p_2)(s - p_3)\dots(s - p_n)} \quad (11.3)$$

The circuits and equations in this textbook will have a denominator with a higher order than the numer-

ator (i.e. there are more poles than zeros). Otherwise, the partial fraction expansion process will require a different process that will not be described in this book.

The partial fraction expansion process converts a factored expression (such as that given in equation 11.3) into a sum of many expressions (shown in equation 11.4, which can then be converted to time-domain expressions using table 11.1.

$$F(s) = \frac{K_1}{(s - p_1)} + \frac{K_2}{(s - p_2)} + \frac{K_3}{(s - p_3)} + \dots + \frac{K_n}{(s - p_n)} \quad (11.4)$$

One possibility will be that poles are real and unique, such as those shown in equation 11.5.

$$F(s) = \frac{K_1}{s + n} + \frac{K_2}{s + m} \quad (11.5)$$

To calculate coefficient K_1 , multiply both sides of the equation by $(s + n)$ and solve at $s = -n$. To calculate coefficient K_2 , multiply both sides of the equation by $(s + m)$ and solve at $s = -m$.

Example: Partial fraction expansion with unique real roots

Find a partial fraction expansion of the following function, then convert into the time domain.

$$F(s) = \frac{4s - 5}{s^2 + 6s + 5}$$

First, factor the denominator.

$$F(s) = \frac{4s - 5}{(s + 1)(s + 5)} = \frac{K_1}{s + 1} + \frac{K_2}{s + 5}$$

The roots are real and unique. Perform a partial fraction expansion and calculate coefficients K_1 and K_2 .

$$\begin{aligned} K_1 &= \left. \frac{4s - 5}{s + 5} \right|_{s=-1} \\ &= \frac{-4 - 5}{-1 + 5} \\ &= -2.25 \\ K_2 &= \left. \frac{4s - 5}{s + 1} \right|_{s=-5} \\ &= \frac{-20 - 5}{-4} \\ &= 6.25 \end{aligned}$$

The expanded version of $F(s)$ is given below.

$$F(s) = \frac{-2.25}{s + 1} + \frac{6.25}{s + 5}$$

Consult table 11.1 to derive the time-domain function.

$$f(t) = [-2.25 e^{-t} + 6.25 e^{-5t}] u(t)$$

A second possibility will be that the poles are real and repeated, such as those shown in equation 11.6.

$$F(s) = \frac{K_1}{(s+n)^2} + \frac{K_2}{s+n} \quad (11.6)$$

To calculate coefficient K_1 , multiply both sides of the equation by $(s+n)^2$ and solve at $s = -n$. To calculate coefficient K_2 , multiply both sides of the equation by $(s+n)^2$, differentiate with respect to s , then solve at $s = -n$.

Example: Partial fraction expansion with repeated real roots

Find a partial fraction expansion of the following function, then convert into the time domain.

$$F(s) = \frac{-2s}{s^2 + 10s + 25}$$

First, factor the denominator.

$$F(s) = \frac{-2s}{(s+5)^2} = \frac{K_1}{(s+5)^2} + \frac{K_2}{s+5}$$

The roots are real and repeated. Perform a partial fraction expansion and calculate coefficients K_1 and K_2 .

$$\begin{aligned} K_1 &= -2s \Big|_{s=-5} \\ &= (-2)(-5) \\ &= 10 \\ \frac{d}{ds} K_2(s+5) \Big|_{s=-5} &= \frac{d}{ds} (-2s) \Big|_{s=-5} \\ &= -2 \end{aligned}$$

The expanded version of $F(s)$ is given below.

$$F(s) = \frac{10}{(s+5)^2} + \frac{-2}{s+5}$$

Consult table 11.1 to derive the time-domain function.

$$f(t) = [10 t e^{-5t} - 2 e^{-5t}] u(t)$$

The third possibility will be that the poles are complex conjugate pairs, such as those shown in equa-

tion 11.7.

$$F(s) = \frac{K_1}{s + \alpha + j\beta} + \frac{K_2}{s + \alpha - j\beta} \quad (11.7)$$

Coefficients K_1 and K_2 will be complex conjugate pairs that take the form of $X \pm jY$. To calculate the coefficients, multiply both sides of the equation by $(s + \alpha - j\beta)$ and solve at $s = -\alpha + j\beta$. Because the coefficients are complex conjugates, the solution takes the form given in equation 11.8. This is derived below using Euler's equations for sine and cosine.

$$f(t) = e^{-\alpha t} [2X \cos(\beta t) + 2Y \sin(\beta t)] u(t) \quad (11.8)$$

DERIVATION

$$\begin{aligned} F(s) &= \frac{X + jY}{s + \alpha + j\beta} + \frac{X - jY}{s + \alpha - j\beta} \\ f(t) &= \left[(X + jY) e^{(-\alpha - j\beta)t} + (X - jY) e^{(-\alpha + j\beta)t} \right] u(t) \\ &= e^{-\alpha t} \left[(X + jY) e^{-j\beta t} + (X - jY) e^{j\beta t} \right] u(t) \\ &= e^{-\alpha t} \left[X e^{-j\beta t} + X e^{j\beta t} + jY e^{-j\beta t} - jY e^{j\beta t} \right] u(t) \\ &= e^{-\alpha t} \left[2X \frac{e^{-j\beta t} + e^{j\beta t}}{2} + 2Y \frac{-e^{-j\beta t} + e^{j\beta t}}{j2} \right] u(t) \\ &= e^{-\alpha t} [2X \cos(\beta t) + 2Y \sin(\beta t)] u(t) \end{aligned}$$

Example: Partial fraction expansion with complex conjugate roots

Find a partial fraction expansion of the following function, then convert into the time domain.

$$F(s) = \frac{-40s + 5200}{s^2 + 40s + 40400}$$

First, factor the denominator.

$$F(s) = \frac{-40s + 5200}{(s + 20 + j200)(s + 20 - j200)} = \frac{K_1}{s + 20 + j200} + \frac{K_2}{s + 20 - j200}$$

The roots are complex conjugate pairs. Perform a partial fraction expansion and calculate coefficient K_1 .

(Note, K_2 is simply the complex conjugate of K_1 , so it is not necessary to calculate it.)

$$\begin{aligned} K_1 &= \left. \frac{-40s + 5200}{s + 20 - j200} \right|_{s=-20-j200} \\ &= \frac{(-40)(-20 - j200) + 5200}{-j400} \\ &= \frac{6000 + j8000}{-j400} \\ &= -20 + j15 \end{aligned}$$

The expanded version of $F(s)$ is given below.

$$F(s) = \frac{-20 + j15}{s + 20 + j200} + \frac{-20 - j15}{s + 20 - j200}$$

Consult equation 11.8 to derive the time-domain function.

$$f(t) = e^{-20t} [-40 \cos(200t) + 30 \sin(200t)] u(t)$$

Example: Partial fraction expansion with mixed roots

Find a partial fraction expansion of the following function, then convert into the time domain.

$$F(s) = \frac{s + 100}{s(s^2 + 400)(s^2 + 6.5s + 10)}$$

First, factor the denominator.

$$F(s) = \frac{s + 100}{s(s + j20)(s - j20)(s + 2.5)(s + 4)} = \frac{K_1}{s} + \frac{K_2}{s + j20} + \frac{K_3}{s - j20} + \frac{K_4}{s + 2.5} + \frac{K_5}{s + 4}$$

K_1 corresponds to a single real root.

$$\begin{aligned} K_1 &= \left. \frac{s + 100}{(s^2 + 400)(s^2 + 6.5s + 10)} \right|_{s=0} \\ &= \frac{100}{4000} \\ &= 0.025 \end{aligned}$$

K_2 and K_3 correspond to complex conjugate roots.

$$\begin{aligned} K_2 &= \left. \frac{s + 100}{s(s - j20)(s^2 + 6.5s + 10)} \right|_{s=-j20} \\ &= \frac{-j20 + 100}{(-j20)(-j40)((-j20)^2 + 6.5(-j20) + 10)} \\ &= 2.7 \times 10^{-4} - j1.5 \times 10^{-4} \end{aligned}$$

K4 and K5 correspond to unique real roots.

$$\begin{aligned}
 K_4 &= \frac{s + 100}{s(s^2 + 400)(s + 4)} \Big|_{s=-2.5} \\
 &= \frac{97.5}{(-2.5)(406.25)(1.5)} \\
 &= -0.064 \\
 K_5 &= \frac{s + 100}{s(s^2 + 400)(s + 2.5)} \Big|_{s=-4} \\
 &= \frac{96}{(-4)(416)(-1.5)} \\
 &= 0.038
 \end{aligned}$$

The expanded version of $F(s)$ is given below.

$$F(s) = \frac{0.025}{s} + \frac{2.7 \times 10^{-4} - j1.5 \times 10^{-4}}{s + j20} + \frac{2.7 \times 10^{-4} + j1.5 \times 10^{-4}}{s - j20} + \frac{-0.064}{s + 2.5} + \frac{0.038}{s + 4}$$

Consult equation 11.8 and table 11.1 to derive the time-domain function.

$$f(t) = [0.025 + 0.00054 \cos(20t) + 0.00031 \sin(20t) - 0.064 e^{-2.5t} + 0.038 e^{-4t}] u(t)$$

11.3 s-Domain Analysis

It is possible to fully analyze a circuit in the s-domain and use the inverse Laplace transform to derive a time-domain function. While this is not always easier than a differential equation analysis, it is frequently more convenient, especially when the equation is non-constant input non-homogeneous, or the circuit is higher than second order.

To perform an s-domain analysis, transform each individual circuit component into the s domain. This is similar to phasor analysis, with the notable difference being that s-domain analysis includes initial conditions in the circuit transformation. Once all circuit elements are transformed into the s-domain, perform any circuit analysis technique to obtain an equation for $V(s)$ or $I(s)$. Then perform an inverse Laplace transform to obtain a time-domain expression.

11.3.1 Resistor

The voltage and current properties of a resistor are defined by Ohm's law (equation 2.19). Converting to the s-domain, the relationship between voltage and current becomes that defined in equation 11.9.

$$V(s) = RI(s) \tag{11.9}$$

Therefore, in the s domain, a resistor is expressed simply as a resistor. This transform is depicted in figure 11.1.

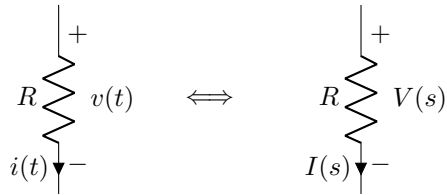


Figure 11.1: Transformation of a resistor from the time domain into the s -domain.

11.3.2 Capacitor

The voltage and current properties of a capacitor are defined in equation 5.5. In the s -domain, this corresponds to the relationship shown in equation 11.10 and derived below.

$$V(s) = \frac{1}{sC}I(s) + \frac{v(0)}{s} \quad (11.10)$$

DERIVATION

$$\begin{aligned} i(t) &= C \frac{dv(t)}{dt} \\ I(s) &= C [sV(s) - v(0)] \\ &= sCV(s) - Cv(0) \\ V(s) &= \frac{1}{sC}I(s) + \frac{v(0)}{s} \end{aligned}$$

Therefore, in the s domain, a capacitor is expressed as a capacitor (with impedance of $(sC)^{-1}$) in series with a voltage source corresponding to the initial conditions of the capacitor. This transform is depicted in figure 11.2.

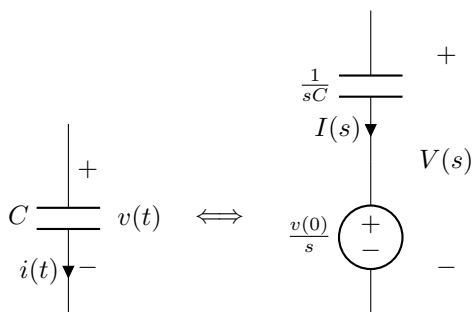


Figure 11.2: Transformation of a capacitor from the time domain into the s -domain.

11.3.3 Inductor

The voltage and current properties of an inductor are defined in equation 6.4. In the s-domain, this corresponds to the relationship shown in equation 11.11 and derived below.

$$I(s) = \frac{1}{sL}V(s) + \frac{i(0)}{s} \quad (11.11)$$

DERIVATION

$$v(t) = L \frac{di(t)}{dt}$$

$$V(s) = L [sI(s) - i(0)]$$

$$= sLI(s) - Li(0)$$

$$I(s) = \frac{1}{sL}V(s) + \frac{i(0)}{s}$$

Therefore, in the s domain, an inductor is expressed as an inductor (with impedance of sL) in parallel with a current source corresponding to the initial conditions of the inductor. This transform is depicted in figure 11.3.

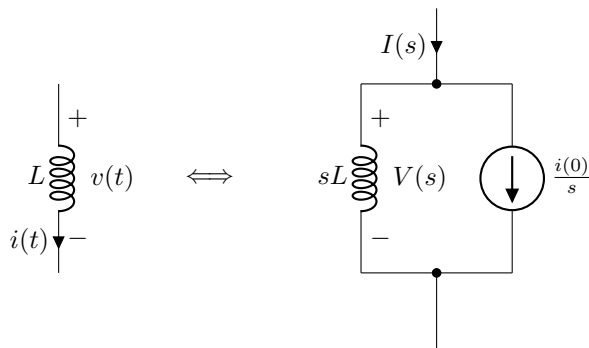


Figure 11.3: Transformation of an inductor from the time domain into the s-domain.

11.3.4 Switched Sources

Frequently, to demonstrate a change in a circuit leading to a transient response, a source will be connected to a circuit via an opening or closing switch. Sources connected via an opening switch (that are disconnected from the steady-state response of the circuit) contribute to initial conditions that are included in the s-domain model of a capacitor (figure 11.2) and inductor (figure 11.3). Sources that are connected via a closing switch do not contribute to the initial conditions, but do contribute to the transient and/or steady-state response. Therefore, a source connected via a closing switch can be represented as $v(t) u(t)$ or $i(t) u(t)$. When transformed into the s-domain, this becomes $V(s)/s$ or $I(s)/s$. This transformation is depicted in figure 11.4 for a voltage source.

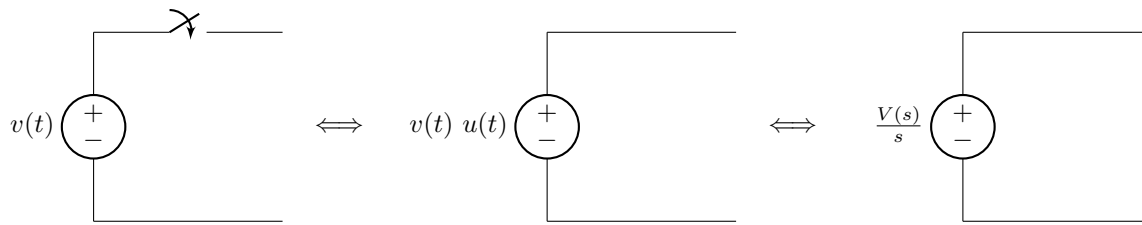
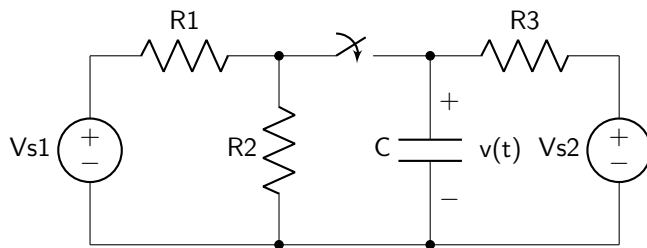


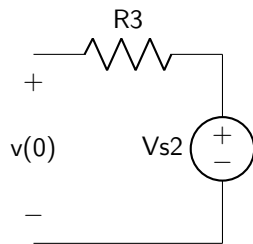
Figure 11.4: Transformation of a switched voltage source from the time domain into the s-domain.

Example: First order circuit s-domain analysis

Calculate $V(s)$ for the circuit given below, and then use the inverse Laplace transform to find an equation for $v(t)$.

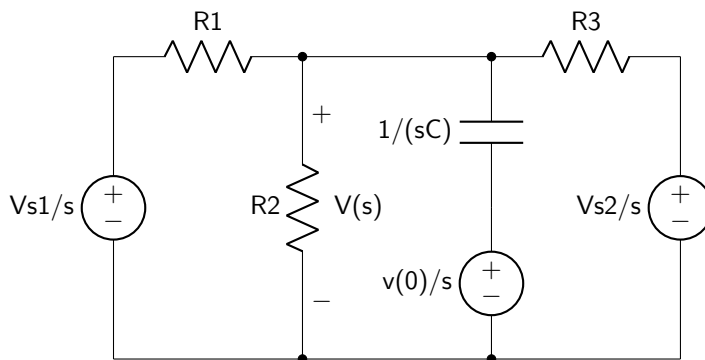


Calculate the initial conditions of the circuit.



$$v(0) = V_{S2}$$

Transform the circuit into the s-domain.



Perform KCL at the node between all three resistors and the capacitor. Group like terms and normalize to find $V(s)$.

$$\begin{aligned}\frac{V_{S1} - V(s)}{R_1} &= \frac{V(s)}{R_2} + \frac{V(s) - \frac{v(0)}{s}}{\frac{1}{sC}} + \frac{V(s) - \frac{V_{S2}}{s}}{R_3} \\ \frac{V_{S1}}{sR_1} &= \frac{V(s)}{R_1} + \frac{V(s)}{R_2} + sCV(s) - \frac{sCv(0)}{s} + \frac{V(s)}{R_3} - \frac{V_{S2}}{sR_3} \\ \frac{V_{S1}}{sR_1} + \frac{V_{S2}}{sR_3} + \frac{sCv(0)}{s} &= sCV(s) + \left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right] V(s) \\ \frac{1}{s} \left[sv(0) + \frac{R_3V_{S1} + R_1V_{S2}}{CR_1R_3} \right] &= sV(s) + \left[\frac{1}{CR_1} + \frac{1}{CR_2} + \frac{1}{CR_3} \right] V(s) \\ V(s) &= \frac{sv(0) + \frac{R_3V_{S1} + R_1V_{S2}}{CR_1R_3}}{s \left(s + \left[\frac{1}{CR_1} + \frac{1}{CR_2} + \frac{1}{CR_3} \right] \right)}\end{aligned}$$

The parameter values are: $V_{S1} = 5 \text{ V}$, $V_{S2} = 10 \text{ V}$, $C = 40 \mu\text{F}$, $R_1 = 5 \Omega$, $R_2 = 10 \Omega$, and $R_3 = 10 \Omega$. Find a numeric expression for $V(s)$.

$$\begin{aligned}V(s) &= \frac{10s + 50000}{s(s + 10000)} \\ &= \frac{K_1}{s} + \frac{K_2}{s + 10000}\end{aligned}$$

Perform a partial fraction expansion to calculate the coefficients.

$$\begin{aligned}K_1 &= \left. \frac{10s + 50000}{s + 10000} \right|_{s=0} \\ &= \frac{50000}{10000} \\ &= 5 \\ K_2 &= \left. \frac{10s + 50000}{s} \right|_{s=-10000} \\ &= \frac{-50000}{-10000} \\ &= 5\end{aligned}$$

The expanded version of $F(s)$ is given below.

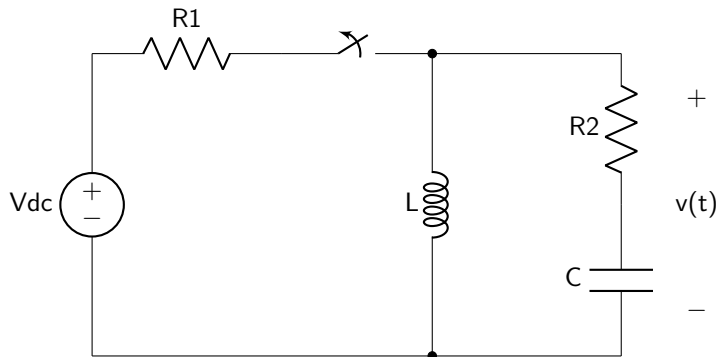
$$F(s) = \frac{5}{s} + \frac{5}{s + 10000}$$

Consult table 11.1 to derive the time-domain function.

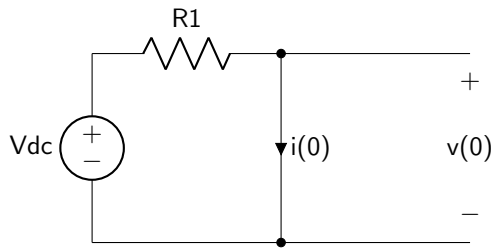
$$f(t) = [5 \text{ V} + 5 \text{ V} e^{-10000t}] u(t)$$

Example: Second order circuit s-domain analysis, symbolic

Calculate $V(s)$ for the circuit given below.



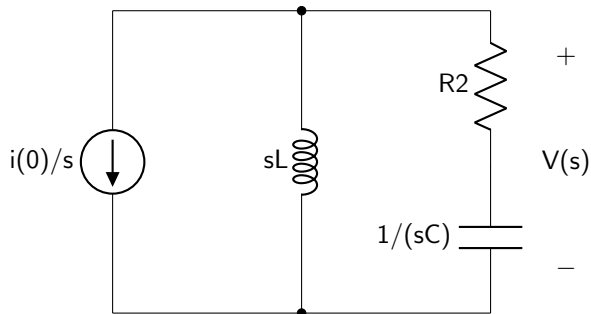
Calculate the initial conditions of the circuit.



$$v(0) = 0$$

$$i(0) = \frac{V_{DC}}{R_1}$$

Transform the circuit into the s-domain.

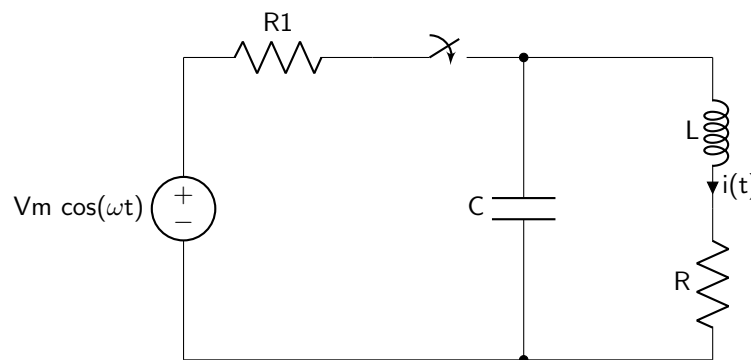


Perform KCL. Group like terms and normalize to find $V(s)$.

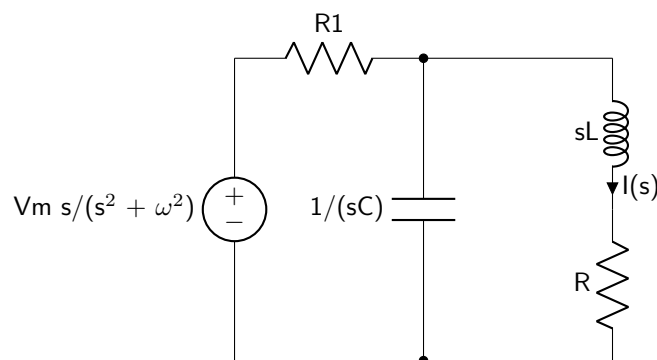
$$\begin{aligned} \frac{-i(0)}{s} &= \frac{V(s)}{sL} + \frac{V(s)}{R_2 + \frac{1}{sC}} \\ \frac{-i(0)}{s} \left[R_2 + \frac{1}{sC} \right] &= \frac{V(s)}{sL} \left[R_2 + \frac{1}{sC} \right] + V(s) \\ \frac{-R_2 i(0)}{s} - \frac{i(0)}{s^2 C} &= \frac{R_2}{sL} V(s) + \frac{1}{s^2 LC} V(s) + V(s) \\ -sR_2 i(0) - \frac{i(0)}{C} &= s^2 V(s) + \frac{R_2}{L} sV(s) + \frac{1}{LC} V(s) \\ V(s) &= \frac{-R_2 i(0)s - \frac{i(0)}{C}}{s^2 + \frac{R_2}{L}s + \frac{1}{LC}} \end{aligned}$$

Example: Second order circuit s-domain analysis, numeric

Calculate $I(s)$ for the circuit given below, and then use the inverse Laplace transform to find an equation for $i(t)$.



The initial conditions are zero, as no source is connected before the switch closes. Transform the circuit into the s-domain.



Perform KCL. Group like terms and normalize to find $I(s)$. (Note that V_x is the voltage dropped over

the capacitor.)

$$\begin{aligned}\frac{\frac{V_m s}{s^2 + \omega^2} - V_X}{R_1} &= sC V_X + I(s) \\ \frac{V_m s}{R_1(s^2 + \omega^2)} &= \frac{V_X}{R_1} + sC V_X + I(s) \\ V_X &= sL I(s) + R_2 I(s) \\ \frac{V_m s}{R_1(s^2 + \omega^2)} &= \frac{1}{R_1} [sL I(s) + R_2 I(s)] + sC [sL I(s) + R_2 I(s)] + I(s) \\ &= LC s^2 I(s) + \left[\frac{L}{R_1} + CR_2 \right] s I(s) + \left[\frac{R_2}{R_1} + 1 \right] I(s) \\ \frac{V_m s}{LC R_1 (s^2 + \omega^2)} &= s^2 I(s) + \left[\frac{1}{CR_1} + \frac{R_2}{L} \right] s I(s) + \left[\frac{R_1 + R_2}{LC R_1} \right] I(s) \\ I(s) &= \frac{\frac{V_m s}{LC R_1}}{(s^2 + \omega^2) \left(s^2 + \left[\frac{1}{CR_1} + \frac{R_2}{L} \right] s + \left[\frac{R_1 + R_2}{LC R_1} \right] \right)}\end{aligned}$$

The parameter values are: $V_m = 0.5 \text{ V}$, $\omega = 50 \text{ rad/s}$, $R_1 = 200 \text{ } \Omega$, $R_2 = 1 \text{ } \Omega$, $C = 50 \text{ } \mu\text{F}$, and $L = 4 \text{ mH}$. Find a numeric expression for $I(s)$.

$$\begin{aligned}I(s) &= \frac{12500s}{(s^2 + 2500)(s^2 + 350s + 5025000)} \\ &= \frac{K_1}{s + j50} + \frac{K_2}{s - j50} + \frac{K_3}{s + 175 + j2234.8} + \frac{K_4}{s + 175 - j2234.8}\end{aligned}$$

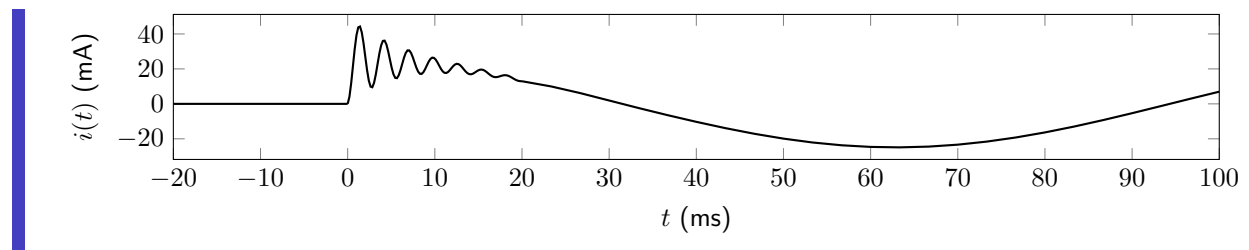
Perform a partial fraction expansion to calculate the coefficients.

$$\begin{aligned}K_1 &= \left. \frac{12500s}{(s - j50)(s^2 + 350s + 5025000)} \right|_{s=-j50} \\ &= \frac{-j625000}{(-j100)(-2500 - j17500 + 5025000)} \\ &= (12.44 + j0.04) \text{ mA} \\ K_3 &= \left. \frac{12500s}{(s^2 + 2500)(s + 175 + j2234.8)} \right|_{s=-175-j2234.8} \\ &= (-12.44 - j0.98) \text{ mA}\end{aligned}$$

The time-domain function is

$$\begin{aligned}i(t) &= [24.89 \text{ mA} \cos(50t) + 0.09 \text{ mA} \sin(50t) \\ &\quad + e^{-175t} (-24.89 \text{ mA} \cos(2234.8t) + 1.95 \text{ mA} \sin(2234.8t))] u(t).\end{aligned}$$

The function is plotted below.



Example Problems

Laplace Transforms

1. Derive the Laplace transform of $f(t) = 2 e^{-5t} u(t)$.
2. Derive the Laplace transform of $f(t) = [6 t e^{-4t} + 2 e^{-4t} + 5] u(t)$.
3. Derive the Laplace transform of $f(t) = e^{-t} [2 \cos(10t) + 4 \sin(10t)] u(t)$.
4. Derive the Laplace transform of $v(t)$. The differential equation describing $v(t)$ is given in equation 11.12. Note that $v(0) = 3$ V.

$$\frac{dv(t)}{dt} + 0.25v(t) = 0 \quad (11.12)$$

5. Derive the Laplace transform of $i(t)$. The differential equation describing $i(t)$ is given in equation 11.13. Note that $i(0) = 10$ A and $i'(0) = -5$ A/s.

$$\frac{d^2i(t)}{dt^2} + 6 \frac{di(t)}{dt} + 109i(t) = 0 \quad (11.13)$$

Inverse Laplace Transforms

6. Derive $f(t)$ by calculating the Laplace transform of $F(s)$, given in equation 11.14. Identify the terms that contribute to the transient response, then identify the terms that contribute to the steady-state response.

$$F(s) = \frac{10}{s + 10} \quad (11.14)$$

7. Derive $f(t)$ by calculating the Laplace transform of $F(s)$, given in equation 11.15. Identify the terms that contribute to the transient response, then identify the terms that contribute to the steady-state response.

$$F(s) = \frac{10s^2 + 10}{s(s^2 + 10s + 25)} \quad (11.15)$$

8. Derive $f(t)$ by calculating the Laplace transform of $F(s)$, given in equation 11.16. Identify the terms that contribute to the transient response, then identify the terms that contribute to the steady-state response.

$$F(s) = \frac{s^2 + 4}{(s^2 + 16)(s^2 + 5s + 4)} \quad (11.16)$$

9. Derive $f(t)$ by calculating the Laplace transform of $F(s)$, given in equation 11.17. Identify the terms that contribute to the transient response, then identify the terms that contribute to the steady-state response.

$$F(s) = \frac{5s^2 + 10s + 3}{(s + 2)^2(s^2 + 5s + 64)} \quad (11.17)$$

10. Derive $f(t)$ by calculating the Laplace transform of $F(s)$, given in equation 11.18. Identify the terms that contribute to the transient response, then identify the terms that contribute to the steady-state response.

$$F(s) = \frac{-4s^2 + 10}{s(s^2 + 25)(s^2 + 20s + 100)} \quad (11.18)$$

s-Domain Analysis

11. Use s-domain analysis to calculate $V(s)$ and $v(t)$ of the circuit shown in figure 11.5. The switch moves from position a to b at a time of zero seconds. The component values are: $V_{S1} = 3$ V, $V_{S2} = 2$ V, $R_1 = 50$ Ω , $R_2 = 100$ Ω , $C = 250$ μ F.

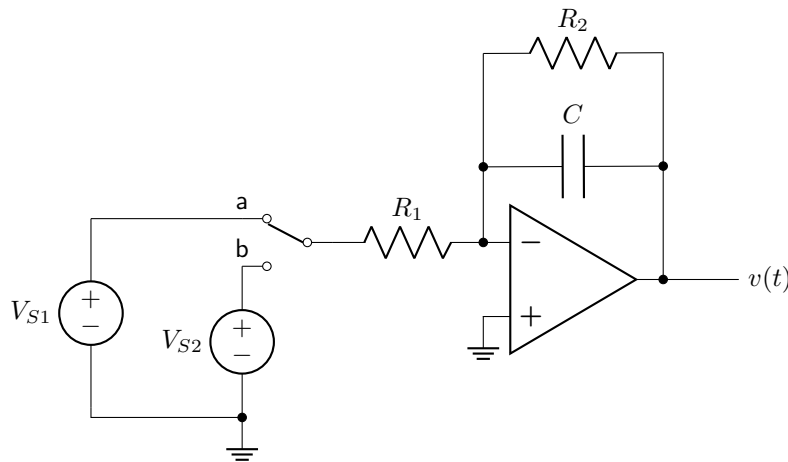


Figure 11.5: Circuit diagram for s-domain analysis question 11.

12. Use s-domain analysis to calculate $V(s)$ of the circuit shown in figure 11.6. The switch closes at a time of zero seconds.

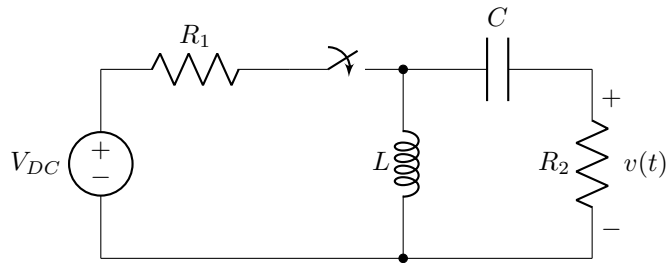


Figure 11.6: Circuit diagram for s-domain analysis question 12.

13. Use s-domain analysis to calculate $V(s)$ and $v(t)$ of the circuit shown in figure 11.7. The switch moves from position a to b at a time of zero seconds. The component values are: $V_{S1} = 6$ V, $V_{S2} = 20$ V, $R_1 = 4$ Ω , $R_2 = 2$ Ω , $C = 50$ μ F, $L = 1$ mH.

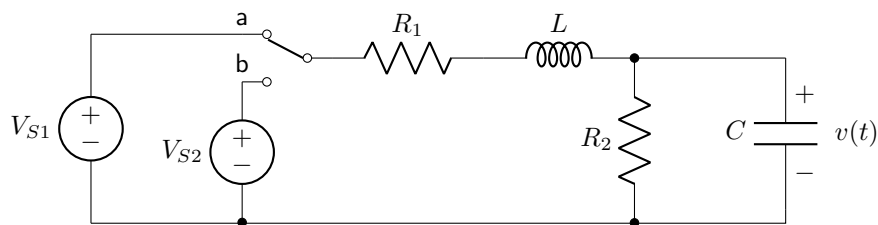


Figure 11.7: Circuit diagram for s-domain analysis question 13.

14. Use s-domain analysis to calculate $I(s)$ and $i(t)$ of the circuit shown in figure 11.8. The switch moves from position b to a at a time of zero seconds. The component values are: $V_{S1} = 40$ V, $\alpha = 4$ rad/s, $V_{S2} = 10$ V, $R = 50$ Ω , $C = 10$ μ F, $L = 400$ mH.

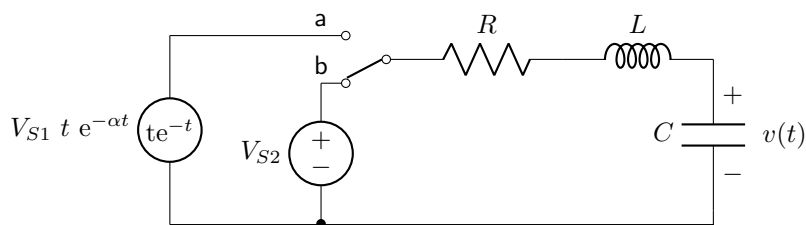


Figure 11.8: Circuit diagram for s-domain analysis question 14.

15. Use s-domain analysis to calculate $I(s)$ and $i(t)$ of the circuit shown in figure 11.9. The switch closes at a time of zero seconds. The component values are: $I_m = 500$ mA, $\omega = 60$ rad/s, $R_1 = 200$ Ω , $R_2 = 2$ Ω , $R_3 = 1$ Ω , $C = 20$ μ F, $L = 50$ mH.

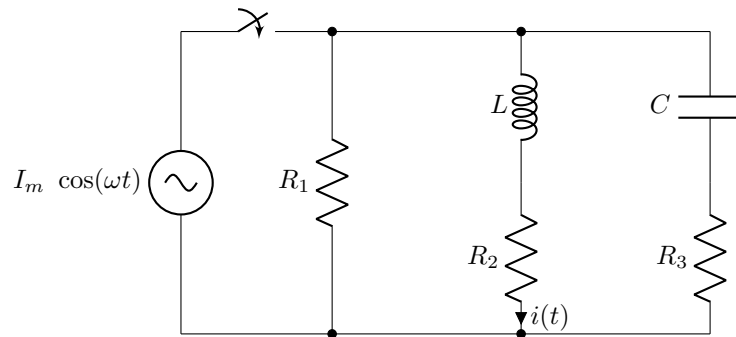


Figure 11.9: Circuit diagram for s-domain analysis question 15.

Solutions to Example Problems

Chapter 1

Power

1. **8 mW** – Use conservation of power to find the unknown power. $10 \text{ mW} + 3 \text{ mW} - 5 \text{ mW} + P = 0$. $P = -8 \text{ mW}$. The negative sign indicates that the circuit element *supplies* 8 mW of power (hence the positive sign in the answer).
2. **416.67 mA** – Use equation 1.2. $i(t) = p(t)/v(t) = 5 \text{ W}/12 \text{ V} = 416.67 \text{ mA}$.
3. **(a) Elements A, B, C (b) Element D** – Current flows from + to – in elements A, B, and C, hence they absorb power. Current flows from – to + in element D, hence it delivers power.
4. **$0.5\cos^2(2\pi 50t)$ W** – Multiply the two functions together.
5. **$120t \delta(t-5)$ mW** – Multiply the two functions together. Units of the solution are in mW because current has units of mA.

Sinusoidal Waves

6. **$V_m = 3.8 \text{ V}$, $V_{DC} = 3 \text{ V}$, $f = 20 \text{ Hz}$, $\phi = 25^\circ$** – Use equation 1.3 to determine the properties of the wave.
7. **$V_m = 2.19 \text{ V}$, $V_{PP} = 4.38 \text{ V}$, $T = 0.025 \text{ s}$** – Use equation 1.3 to determine the properties of the wave.
8. **$V_m = 2 \text{ V}$, $V_{DC} = 1 \text{ V}$, $f = 125 \text{ Hz}$** – Use figure 1.11 to determine the properties of the wave, and note from equation 1.3 that $f = 1/T$.
9. **The waveform is shown in figure SOL.1.**

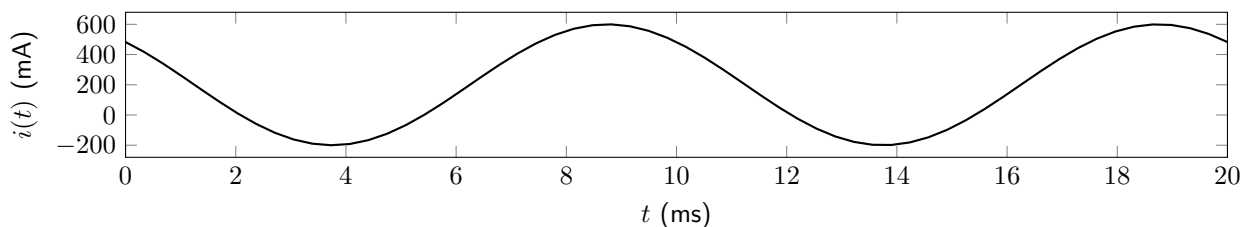


Figure SOL.1: Waveform for sinusoidal waves answer 9.

10. **30 mA** – The DC offset required to remove negative components is equal to the magnitude.

Sources

11. **VCVS.** The controlling value is a voltage, and the source itself is a voltage source.
12. **CCVS.** The controlling value is a current, and the source itself is a voltage source.
13. **-0.96 mA** – The current measured by the ammeter is 4.8 mA, which is opposite the direction of I_X as defined in the circuit diagram. Therefore $I_X = -4.8$ mA. Multiply by the proportionality constant of 0.2 to calculate the magnitude of the dependent current.
14. **The circuit diagram is shown in figure SOL.2.**

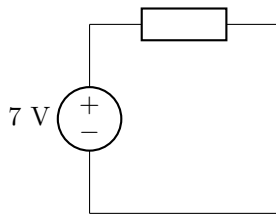


Figure SOL.2: Circuit schematic for sources answer 14.

15. **The circuit diagram is shown in figure SOL.3.**

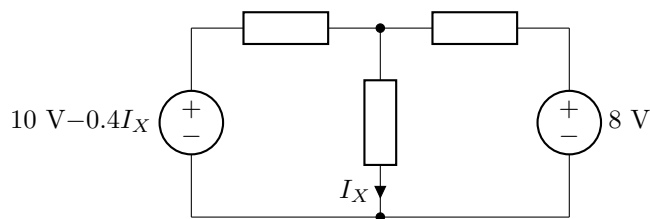


Figure SOL.3: Circuit schematic for sources answer 15.

Elementary Signals

16. $f(t) = 2\delta(t+4) - \delta(t+2) - \delta(t) + 1.5\delta(t-5)$
17. **There are two possible answers, depending on which type of elementary signal is used to find the answer.** – Using the step function, $f(t) = u(t+4) + u(t+1) - 3u(t-1) + u(t-4)$. Using the rectangular pulse function, $f(t) = \text{rect}\left(\frac{t+2.5}{3}\right) + 2\text{rect}\left(\frac{t}{2}\right) - \text{rect}\left(\frac{t-2.5}{3}\right)$.

18. The signal is shown in figure SOL.4.

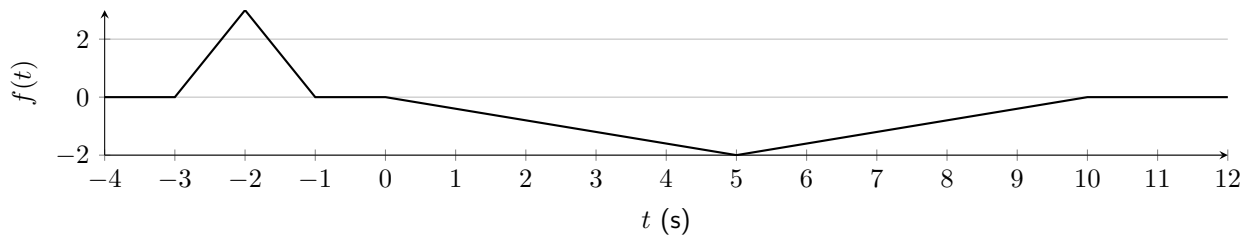


Figure SOL.4: Signal function for elementary signals answer 18.

19. The signal is shown in figure SOL.5.

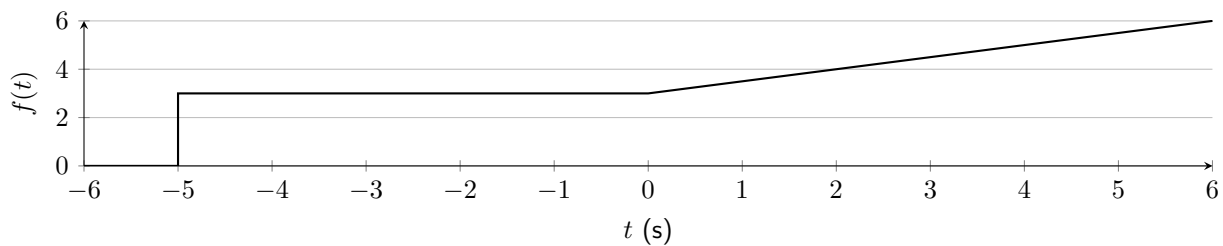


Figure SOL.5: Signal function for elementary signals answer 19.

20. The current is shown in figure SOL.6.

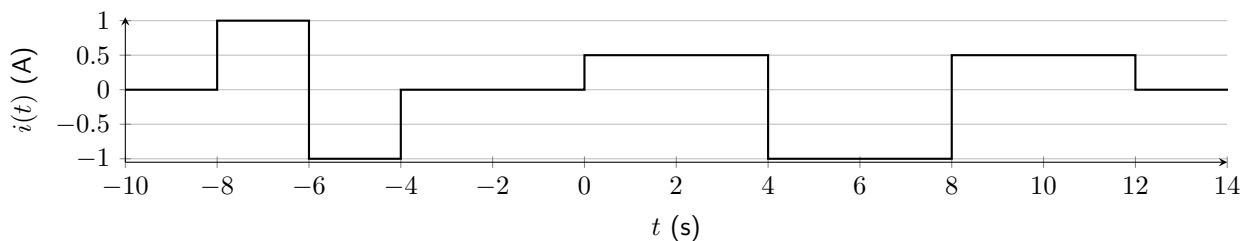


Figure SOL.6: Signal function for elementary signals answer 20.

Chapter 2

Equivalent Resistance

1. The equivalent resistance is given in equation SOL.1. – Combine R_2 and R_3 in parallel. Then combine that result in series with R_1 .

$$R_{EQ} = R_1 + \frac{R_2 R_3}{R_2 + R_3} \quad (\text{SOL.1})$$

2. **The reduced circuit diagram is shown in figure SOL.7.** – The only two resistors that can be combined are the $10\ \Omega$ and $24\ \Omega$ resistors. No other resistors are in purely series or purely parallel combinations with each other.

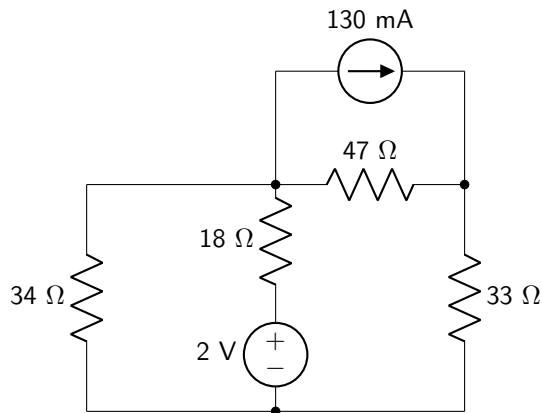


Figure SOL.7: Circuit diagram for equivalent resistance answer 2.

3. **1400 Ω** – Note that you will have to perform either a delta-wye or wye-delta transform to solve this question.
4. **733.333 Ω** – This calculation only requires series and parallel combinations.
5. **20.448 k Ω** – Note that you will have to perform either a delta-wye or wye-delta transform to solve this question.

Ohm's Law

6. **3.7 Ω** – The minimum equivalent resistance of the circuit is $5\ \Omega$.
7. **-3.125 mA** – Because the current flow is defined opposite to the direction of the voltage drop, the current is going to be a negative number.
8. **51.2 μW** – Using the current divider rule, the current flowing through the $20\ \Omega$ resistor is calculated to be $1.6\ \text{mA}$. Then, use equation 2.23 to calculate the power consumed by the resistor. Be careful with the units. When mA are squared, the resulting units are μA^2 .
9. **234.375 W** – The equivalent resistance of the circuit is $9.375\ \Omega$. Then use equation 2.23 to calculate the power absorbed by the resistor, which must be equal to the power supplied by the source.
10. **400 Ω** – Use the voltage divider rule to determine that the voltage dropped over each resistor will be $10\ \text{V}$. Then set equation 2.24 equal to $0.25\ \text{W}$ to calculate the minimum allowable value of R .

Voltage and Current Divider

11. $4.8 \text{ V} - V_X = 7.2 \text{ V} \left(\frac{6 \text{ k}\Omega}{3 \text{ k}\Omega + 6 \text{ k}\Omega} \right)$
12. $140 \text{ mA} - I_X = 500 \text{ mA} \left(\frac{42 \text{ }\Omega // 12 \text{ }\Omega // 24 \text{ }\Omega}{24 \text{ }\Omega} \right)$
13. $3.214 \text{ V} - V_X = 8 \text{ V} \left(\frac{470 \text{ }\Omega}{100 \text{ }\Omega + 470 \text{ }\Omega} \right) \left(\frac{570 \text{ }\Omega // 330 \text{ }\Omega}{220 \text{ }\Omega + 570 \text{ }\Omega // 330 \text{ }\Omega} \right)$
14. $93.986 \text{ mA} - I_X = 210 \text{ mA} \left(\frac{200 \text{ }\Omega // 150 \text{ }\Omega}{150 \text{ }\Omega} \right) \left(\frac{400 \text{ }\Omega // (25 \text{ }\Omega + 200 \text{ }\Omega // 150 \text{ }\Omega)}{25 \text{ }\Omega + 200 \text{ }\Omega // 150 \text{ }\Omega} \right)$
15. V_{X1} is given in equation SOL.2, and V_{X2} is given in equation SOL.3.

$$V_{X1} = V_S \left(\frac{R_3}{R_1 + R_2 + R_3} \right) \quad (\text{SOL.2})$$

$$V_{X2} = V_S \left[\frac{R_7}{R_6 + R_7} \right] \left[\frac{R_5 // (R_6 + R_7)}{R_4 + R_5 // (R_6 + R_7)} \right] \quad (\text{SOL.3})$$

Kirchhoff's Laws

16. $3 \text{ V} - 5 \text{ V} - 2 \text{ V} - V_X = 0$
17. $6 \text{ mA} - 10 \text{ mA} - 4 \text{ mA} - I_X = 0$
18. 6.667 mA – There are many different ways to label and solve this question using KCL and KVL. One possible set of equations is $I_1 + I_2 - I_X = 0$; $5 = I_1 + 0.1I_X$; $3 = I_2 + 0.1I_X$ (all units are in V, mA, or k Ω).
19. 1.081 V – There are many different ways to label and solve this question using KCL and KVL. One possible set of equations is $I_1 - I_2 - I_3 = 0$; $10 = 0.1I_1 + 0.05I_2$; $-2 = -0.05I_2 + 1.2I_3$ (all units are in V, mA, or k Ω). Then use Ohm's law to calculate V_X .
20. 120 V – There are many different ways to label and solve this question using KCL and KVL. One possible set of equations is $-0.8I_X + I_1 = 0$; $320 = 100I_1 + 48I_X$ (all units are in V, mA, or k Ω). Then use Ohm's law to calculate V_X .

Mesh Analysis

21. $I_A = 10 \text{ mA}$, $I_B = 8 \text{ mA}$ – Mesh I_A contains a current source, causing I_A to be equal to the value of the current source. The branch current is equal to $I_A - I_B$.
22. 9 A – The branch current is equal to the left mesh current minus the right mesh current.
23. $I_X = -I_2 - I_4 - I_5 - I_7$ – Note that all of the numbered mesh currents are also branch currents.
24. 1.081 V – This answer is and should be identical to the Kirchhoff's laws question. There are many different ways to label and solve this question using mesh analysis. One possible set of equations is

$0.1I_1 + 0.05(I_1 - I_2) = 10$; $0.05(I_2 - I_1) + 1.2I_2 = -2$ (all units are in V, mA, or $k\Omega$). Then use Ohm's law to calculate V_X .

25. **12.181 V** – There are many different ways to label and solve this question using mesh analysis. One possible set of equations is $10I_1 + 20(I_1 - I_2) = 20$; $I_2 + 0.008 - I_3 = 0$; $20(I_2 - I_1) + 60I_2 + 10I_3 = 0$ (all units are in V, A, or Ω). Then use $V_X = 20 \Omega(I_1 - I_2)$.

Chapter 3

Superposition

- 12.72 V** – The voltage contributed by the voltage source is 7.66 V, and the voltage contributed by the current source is 5.06 V.
- 3.64 V** – The voltage contributed by the voltage source is -7 V, and the voltage contributed by the current source is 3.36 V.
- 3.44 mA** – The current contributed by the current source is 2.37 mA, the current contributed by the 24 V voltage source is 0.79 mA, and the current contributed by the 5 V voltage source is 0.28 mA.
- 50 V** – The voltage contributed by the voltage source is 20 V, and the voltage contributed by the current source is 30 V.
- 4.11 V** – The voltage contributed by the current source is 0.32 V, and the voltage contributed by the voltage source is 3.79 V.

Source Transformation

- 12.72 V** – This circuit can be reduced to a single current source of $55^{1/3}$ mA in parallel with a resistor of value 229.895 Ω . (This answer should be and is identical to question 1 in the superposition section.)
- 3.64 V** – This circuit can be reduced to a single voltage source of 10.4 V in series with a 1300 Ω resistor, and the 700 Ω resistor over which $-V_X$ is measured. (This answer should be and is identical to question 2 in the superposition section.)
- 3.44 mA** – This circuit can be reduced to a single voltage source of 62.6 V in series with a resistor of value 18.2 $k\Omega$, through which I_X is measured. (This answer should be and is identical to question 3 in the superposition section.)
- 50 V** – This circuit can be reduced to a single current source of $3^{1/3} + 0.1V_X$ A in parallel with a 6 Ω resistor, over which V_X is measured. (This answer should be and is identical to question 4 in the superposition section.)

10. $\mathbf{V_X = -2.941\ V}$, $\mathbf{V_Y = -2.353\ V}$ – This circuit can be reduced to a single voltage source of $3^{1/3} + 2^{2/3} V_X + 3V_Y$ V in series with a $6^{2/3}\ \Omega$ resistor, a $40\ \Omega$ resistor over which V_Y is measured, and a $50\ \Omega$ resistor over which V_X is measured. Use a matrix to solve for both unknown quantities.

Thévenin and Norton's Theorems

11. $\mathbf{V_{TH} = -3\ V}$, $\mathbf{R_{TH} = 21\ \Omega}$ – Calculate the open-circuit voltage using a voltage divider (and be careful with the sign), then deactivate the voltage source to calculate the equivalent resistance between nodes a and b.
12. $\mathbf{I_N = 28\ mA}$, $\mathbf{R_N = 891.892\ \Omega}$ – Use source transformation or superposition to solve. Deactivate all sources to calculate the equivalent resistance.
13. $\mathbf{V_{TH} = 6.75\ V}$, $\mathbf{R_{TH} = 375\ \Omega}$ – Calculate the open-circuit voltage, then short nodes a and b to calculate a short-circuit current of 18 mA. Use Ohm's law to calculate the Thévenin equivalent resistance.
14. $\mathbf{V_{TH} = 6.2\ V}$, $\mathbf{R_{TH} = 260\ \Omega}$ – Use any circuit analysis tool to calculate the open-circuit voltage and short-circuit current (23.846 mA). Use Ohm's law to calculate the Thévenin equivalent resistance.
15. $\mathbf{I_N = 0.04\ A}$, $\mathbf{R_N = 33.333\ \Omega}$ – Use Ohm's law on both sides of the circuit to calculate V_X and I_X ; V_X is the open-circuit voltage. Then short terminals a and b and use Ohm's law again to calculate I_N . Use Ohm's law to calculate the Norton equivalent resistance.

Maximum Power Transfer

16. $\mathbf{R_{LOAD} = 21\ \Omega}$, $\mathbf{P_{MAX} = 0.107\ W}$ – Use the Thévenin equivalent resistance from question 11, and calculate the maximum power using equation 3.8.
17. $\mathbf{R_{LOAD} = 891.892\ \Omega}$, $\mathbf{P_{MAX} = 0.175\ W}$ – Calculate the Thévenin equivalent resistance from question 12, and calculate the maximum power using equation 3.8.
18. $\mathbf{R_{LOAD} = 375\ \Omega}$, $\mathbf{P_{MAX} = 0.030\ W}$ – Use the Thévenin equivalent resistance from question 13, and calculate the maximum power using equation 3.8.
19. $\mathbf{R_{LOAD} = 260\ \Omega}$, $\mathbf{P_{MAX} = 0.370\ W}$ – Use the Thévenin equivalent resistance from question 14, and calculate the maximum power using equation 3.8.
20. $\mathbf{R_{LOAD} = 33.333\ \Omega}$, $\mathbf{P_{MAX} = 0.013\ W}$ – Calculate the Thévenin equivalent resistance from question 15, and calculate the maximum power using equation 3.8.

Chapter 4

1. $V_{OUT} = -3 \text{ V}$, $A = -6$ – This is an inverting op-amp circuit.
2. $V_{OUT} = 4 \text{ V}$ – This is a difference amplifier circuit. Calculate the voltage at the non-inverting input by using a voltage divider. Then use Ohm's law on the feedback path to calculate the output voltage.
3. $V_{OUT} = 1.5 \text{ V}$, $A = 3$ – Find two equations for two unknowns (the output voltage and the node voltage at the op-amp inputs).
4. $A = 1.5$ – There are three unknowns in this circuit: V_{OUT} , the virtual node voltage of the left-most op-amp, and the output of the left-most op-amp. Use the op-amp properties, Ohm's law, and KCL to find three equations for these three unknowns, and solve for the gain.
5. $V_{OUT} = 4 \text{ V}$ – There are three unknowns in this circuit: V_{OUT} , the virtual node voltage of the left-most op-amp, and the output of the left-most op-amp. Use the op-amp properties, Ohm's law, and KCL to find three equations for these three unknowns, and solve for the gain.

Chapter 5

Capacitance and Equivalent Capacitance

1. $v(t) = 80 \text{ V} [1 - e^{-500t}] u(t)$, $p(t) = 0.08 \text{ W} [e^{-500t} - e^{-1000t}] u(t)$ – To calculate the voltage, take the integral of the current. Use equation 1.2 to calculate the instantaneous power.
2. $i(t) = 0.033 \text{ A} u(t)$, $p(t) = 3.3 \text{ W/s } t u(t)$ – To calculate the current, take the derivative of the voltage. Use equation 1.2 to calculate the instantaneous power.
3. 67.2 pF – To calculate series combinations of capacitors, use equation 5.7. To calculate parallel combinations of capacitors, use equation 5.8.
4. 6.38 nF – To calculate series combinations of capacitors, use equation 5.7. To calculate parallel combinations of capacitors, use equation 5.8.
5. $2 \text{ } \mu\text{F}$ – To calculate series combinations of capacitors, use equation 5.7. To calculate parallel combinations of capacitors, use equation 5.8.

Resistor-Capacitor Circuits

6. $v(t) = 200 \text{ V} e^{-62.5t} u(t) + 200 \text{ V} u(-t)$, $i(t) = -5 \text{ mA} e^{-62.5t} u(t)$ – This is a discharging circuit. Analyze the circuit in the initial steady-state configuration (switch is in position a) to find $v(0)$. Then find the equivalent resistance seen by the capacitor in the final steady-state configuration (switch is in position b) to calculate τ . Use equation 5.12 to calculate $v(t)$ and equation 5.13 to calculate $i(t)$.

7. $v(t) = 8 \text{ V } e^{-5.882t} u(t) + 8 \text{ V } u(-t)$, $i(t) = -0.424 \text{ mA } e^{-5.882t} u(t)$ – This is a discharging circuit. Analyze the circuit in the initial steady-state configuration (switch is closed) to find $v(0)$. Then find the equivalent resistance seen by the capacitor in the final steady-state configuration (switch is open) to calculate τ . Use equation 5.12 to calculate $v(t)$ and equation 5.13 to calculate $i(t)$.
8. $v(t) = [4.44 \text{ V} - 4.44 \text{ V } e^{-90t}] u(t)$, $i(t) = 2 \text{ mA } e^{-90t} u(t)$ – This is a charging circuit. Analyze the circuit in the final steady-state configuration (switch is closed) to find $v(\infty)$ as well as the equivalent resistance seen by the capacitor to calculate τ . Use equation 5.15 to calculate $v(t)$ and equation 5.16 to calculate $i(t)$.
9. $v(t) = [16.875 \text{ V} - 4.875 \text{ V } e^{-6.849t}] u(t) + 12 \text{ V } u(-t)$, $i(t) = 1.336 \text{ mA } e^{-6.849t} u(t)$ – This is a general circuit. Analyze the circuit in the initial steady-state configuration (switch is open) to find $v(0)$. Then analyze the circuit in the final steady-state configuration (switch is closed) to find $v(\infty)$ as well as the equivalent resistance seen by the capacitor. Use equation 5.17 to calculate $v(t)$ and equation 5.18 to calculate $i(t)$.
10. $v(t) = [4 \text{ V} + 10 \text{ V } e^{-1000t}] u(t) + 14 \text{ V } u(-t)$, $i(t) = -0.4 \text{ mA } e^{-1000t} u(t)$ – This is a general circuit. Analyze the circuit in the initial steady-state configuration (switch is closed) to find $v(0)$. Then analyze the circuit in the final steady-state configuration (switch is open) to find $v(\infty)$ as well as the equivalent resistance seen by the capacitor. Use equation 5.17 to calculate $v(t)$ and equation 5.18 to calculate $i(t)$.

Chapter 6

Inductance and Equivalent Inductance

1. $v(t) = u(t) \text{ mV}$, $p(t) = 0.05 \text{ mW } u(t)$ – To calculate the voltage, use equation 6.4. Use equation 1.2 to calculate the instantaneous power.
2. $i(t) = 13.333 \text{ A } \sin(5000t) u(t)$, $p(t) = 6.667 \text{ W } \sin(10000t) u(t)$ – To calculate the current, use equation 6.5. Use equation 1.2 to calculate the instantaneous power.
3. **74.375 mH** – To calculate series combinations of inductors, use equation 6.6. To calculate parallel combinations of inductors, use equation 6.7.
4. **102.24 μH** – To calculate series combinations of inductors, use equation 6.6. To calculate parallel combinations of inductors, use equation 6.7.
5. **300 μH** – To calculate series combinations of inductors, use equation 6.6. To calculate parallel combinations of inductors, use equation 6.7.

Resistor-Inductor Circuits

6. $\mathbf{i(t) = 5\text{ mA } e^{-5000t} \mathbf{u(t) + 5\text{ mA } u(-t)}$, $\mathbf{v(t) = -8\text{ V } e^{-5000t} \mathbf{u(t)}$ – This is a discharging circuit. Analyze the circuit in the initial steady-state configuration (switch is in position a) to find $i(0)$. Then find the equivalent resistance seen by the inductor in the final steady-state configuration (switch is in position b) to calculate τ . Use equation 6.9 to calculate $i(t)$ and equation 6.10 to calculate $v(t)$.
7. $\mathbf{i(t) = 6\text{ mA } e^{-7500t} \mathbf{u(t) + 6\text{ mA } u(-t)}$, $\mathbf{v(t) = -0.9\text{ V } e^{-7500t} \mathbf{u(t)}$ – This is a discharging circuit. Analyze the circuit in the initial steady-state configuration (switch is closed) to find $i(0)$. Then find the equivalent resistance seen by the inductor in the final steady-state configuration (switch is open) to calculate τ . Use equation 6.9 to calculate $i(t)$ and equation 6.10 to calculate $v(t)$.
8. $\mathbf{i(t) = 0.5\text{ mA } [1 - e^{-80000t}] \mathbf{u(t)}$, $\mathbf{v(t) = 4\text{ V } e^{-80000t} \mathbf{u(t)}$ – This is a charging circuit. Analyze the circuit in the final steady-state configuration (switch is closed) to find $i(\infty)$ as well as the equivalent resistance seen by the inductor to calculate τ . Use equation 6.12 to calculate $i(t)$ and equation 6.13 to calculate $v(t)$.
9. $\mathbf{i(t) = [3.143\text{ mA} - 0.643\text{ mA } e^{-35000t}] \mathbf{u(t) + 2.5\text{ mA } u(-t)}$, $\mathbf{v(t) = 5.4\text{ V } e^{-35000t} \mathbf{u(t)}$ – This is a general circuit. Analyze the circuit in the initial steady-state configuration (switch is open) to find $i(0)$. Then analyze the circuit in the final steady-state configuration (switch is closed) to find $i(\infty)$ as well as the equivalent resistance seen by the inductor. Use equation 6.14 to calculate $i(t)$ and equation 6.15 to calculate $v(t)$.
10. $\mathbf{i(t) = [5\text{ mA} - 3\text{ mA } e^{-20000t}] \mathbf{u(t) + 2\text{ mA } u(-t)}$, $\mathbf{v(t) = 6\text{ V } e^{-20000t} \mathbf{u(t)}$ – This is a general circuit. Analyze the circuit in the initial steady-state configuration (switch S1 is closed and switch S2 is open) to find $i(0)$. Then analyze the circuit in the final steady-state configuration (switch S1 is open and switch S2 is closed) to find $i(\infty)$ as well as the equivalent resistance seen by the inductor. Use equation 6.14 to calculate $i(t)$ and equation 6.15 to calculate $v(t)$.

Chapter 7**Homogeneous Second Order Circuits**

1. $\mathbf{v(t) = -50\text{ V/s } t e^{-200t} \mathbf{u(t)}$ – This is a critically-damped series RLC circuit.
2. $\mathbf{i(t) = [0.117\text{ A } e^{-8612.667t} - 0.017\text{ A } e^{-58053.987t}] \mathbf{u(t) + 0.100\text{ A } u(-t)}$ – This is an overdamped parallel RLC circuit.
3. $\mathbf{v(t) = [315.83\text{ mV } e^{-37.94t} - 91.83\text{ mV } e^{-67.77t}] \mathbf{u(t) + 224\text{ mV } u(-t)}$ – This is an overdamped parallel RLC circuit with parasitic resistance in the inductor.
4. $\mathbf{v(t) = e^{-4900t} [1.875\text{ V } \cos(5999.171t) + 1.53\text{ V } \sin(5999.171t)] \mathbf{u(t) + 1.875\text{ V } u(-t)}$ – This is an underdamped general RLC circuit.

5. **The differential equation is given in equation SOL.4.** – This is a general RLC circuit.

$$\frac{d^2 i(t)}{dt^2} + \left[\frac{R_2}{L_1} + \frac{R_3}{L_2} + \frac{R_2}{L_2} + \frac{R_1}{L_1} \right] \frac{di(t)}{dt} + \left[\frac{R_2 R_3}{L_1 L_2} + \frac{R_1 R_3}{L_1 L_2} + \frac{R_1 R_2}{L_1 L_2} \right] i(t) = 0 \quad (\text{SOL.4})$$

Non-Homogeneous Second Order Circuits

6. $v(t) = [e^{-320000t} (6 \text{ V} \cos(240000t) + 3.83 \text{ V} \sin(240000t)) + 20 \text{ V}] u(t) + 26 \text{ V} u(-t)$ – This is an underdamped series RLC circuit.

7. $i(t) = [e^{-640t} (-0.2 \text{ A} \cos(480t) - 0.267 \text{ A} \sin(480t)) + 0.2 \text{ A}] u(t)$ – This is an underdamped parallel RLC circuit.

8. $v(t) = [20 \text{ V} - 21.373 \text{ V} e^{-11567.72t} + 1.373 \text{ V} e^{-180098.94t}] u(t)$ – This is an overdamped general second order circuit.

9. $i(t) = [0.5 \text{ A} + 0.971 \text{ A} e^{-782.099t} - 0.271 \text{ A} e^{-1917.901t}] u(t) + 1.2 \text{ A} u(-t)$ – This is an overdamped general second order circuit.

10. $v(t) = [-0.08 \text{ V} \sin(2\pi 40t) + 0.64 \text{ V} e^{-5t} \sin(31.22t)] u(t)$ – This is a variable input non-homogeneous second order circuit. Start by deriving the second order differential equation for the circuit, shown in equation SOL.5. Then, find the particular solution to the equation. Finally, derive the solution to the homogeneous equation (which is underdamped).

$$5000 \sin(2\pi 40t) = \frac{d^2 v(t)}{dt^2} + 10 \frac{dv(t)}{dt} + 1000v(t) \quad (\text{SOL.5})$$

Chapter 8

Phasor Arithmetic

1. $223.61 \angle -153.43^\circ$
2. $-2.5 - j4.33$
3. $-1315 - j140$ in Cartesian form, $1322 \angle -173.92^\circ$ in polar form
4. $74.73 + j266.24$ in Cartesian form, $276.53 \angle 74.32^\circ$ in polar form
5. $79.13 - j1027.49$ in Cartesian form, $1030.54 \angle -85.60^\circ$ in polar form

Impedance and Equivalent Impedance

6. $Z = j9.425 \Omega$ – Use equation 8.18.
7. $Z = -j169.314 \Omega$ – Use equation 8.17.

8. $\mathbf{Z} = 103.217 + j16.101 \ \Omega$ – Start by calculating the impedance of the inductor. Then combine impedances in series and parallel as required.
9. $\mathbf{Z} = 23.970 + j282.647 \ \Omega$ – Start by calculating the impedance of both inductors and the capacitor. Then combine impedances in series and parallel as required.
10. $\mathbf{Z} = 0.059 - j0.528 \ \Omega$ – Start by calculating the impedance of both capacitors and the inductor. Then combine impedances in series and parallel as required.

Delta-Wye and Wye-Delta Transforms

11. The equivalent circuit is shown in figure SOL.8 – Use equations 8.30–8.32 to complete the transform.

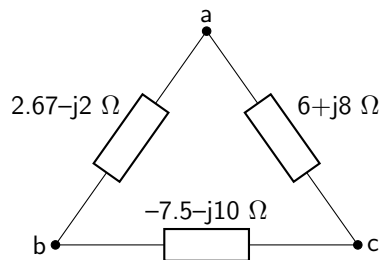


Figure SOL.8: Circuit diagram for delta-wye and wye-delta transforms answer 11.

12. The equivalent circuit is shown in figure SOL.9 – Use equations 8.27–8.28 to complete the transform.

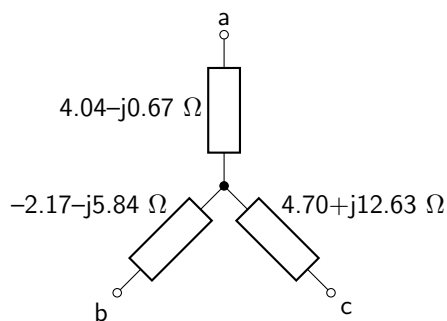


Figure SOL.9: Circuit diagram for delta-wye and wye-delta transforms answer 12.

13. $\mathbf{Z}_{EQ} = 2.86 - j2.17 \ \Omega$ – Use either a delta-wye or a wye-delta transform. The equivalent circuit consists of a $2.86 \ \Omega$ resistor in series with a capacitor of unknown value (the value of the capacitor is unknown because the frequency of operation is not specified).
14. $\mathbf{Z}_{EQ} = 149.15 - j134.42 \ \Omega$ – First, convert all values to impedances. Then use either a delta-wye or a wye-delta transform and combine impedances in series and parallel as needed. The equivalent circuit

consists of a 149.15Ω resistor in series with a 118.40 nF capacitor.

15. $\mathbf{Z}_{\text{EQ}} = 53.10 - j22.89 \Omega$ – First, convert all values to impedances. Then use either a delta-wye or a wye-delta transform and combine impedances in series and parallel as needed. The equivalent circuit consists of a 53.10Ω resistor in series with a $139.05 \mu\text{F}$ capacitor.

Chapter 9

Complex Voltage and Current Divider

- $\mathbf{v(t)} = 41.74 \text{ V} \cos(2\pi 60t + 26.59\pi/180) - \mathbf{V} = (25 + j43.3 \text{ V}) \frac{-j75.79 \Omega}{50 - j75.79 \Omega}$ Start by converting the capacitance to an impedance. Then apply the voltage divider rule.
- $\mathbf{v(t)} = 175.00 \text{ V} \cos(2\pi 80t + 140.10\pi/180) - \mathbf{V} = (-138.56 + j75 \text{ V}) \frac{120 - j75.8 \Omega}{120 - j49.4 \Omega}$ Apply the voltage divider rule.
- $\mathbf{i(t)} = 22.14 \text{ A} \cos(2\pi 150t + 168.43\pi/180) - \mathbf{I} = (-10 - j17.32 \text{ A}) \frac{7.35 - j4.55 \Omega}{6 + j5 \Omega}$ Apply the current divider rule.
- $\mathbf{v(t)} = 3.25 \text{ V} \cos(2\pi 100t - 63.88\pi/180) - \mathbf{V} = (7 \text{ V}) \frac{16.50 - j14.93 \Omega}{44.50 + j17.75 \Omega}$ Start by converting the capacitance and inductance to impedances. Then apply the voltage divider rule.
- $\mathbf{i(t)} = 3.92 \text{ A} \cos(2\pi 6000t + 59.40\pi/180) - \mathbf{I} = (3.90 + j2.25 \text{ A}) \frac{30.36 + j17.11 \Omega}{40 \Omega}$ Start by converting the capacitance and inductance to impedances. Then apply the current divider rule.

Complex Kirchhoff's Laws

- $\mathbf{i(t)} = 57.91 \text{ mA} \cos(2\pi 60t - 75.86\pi/180) -$ There are many different ways to label and solve this question using KCL and KVL. One possible set of equations is $I_1 - I_2 - I_3 = 0$; $24 = 40I_1 + (-j8.04)I_2 + (80 + j3.77)I_3$; $0 = (j8.04)I_2 + (80 + j3.77)I_3$ (all units are in V, A, or Ω).
- $\mathbf{v(t)} = 42.39 \text{ V} \cos(2\pi 20t + 137.47\pi/180) -$ There are many different ways to label and solve this question using KCL and KVL. One possible set of equations is $I_1 - I_2 - I_3 = 0$; $I_3 - I_4 - I_5 = 0$; $-40 + j69.28 = 8I_1 + 10I_2$; $0 = -10I_2 + 5I_3 - j6I_4$; $0 = j6I_4 + j5I_5$ (all units are in V, A, or Ω). Then use Ohm's law to calculate $v(t)$.
- $\mathbf{v(t)} = 27.60 \text{ V} \cos(2\pi 20000t - 53.82\pi/180) -$ There are many different ways to label and solve this question using KCL and KVL. One possible set of equations is $I_1 + I_2 = 15$; $I_2 - I_3 - I_4 = 0$; $0 = -6I_1 + j9I_2 + (4 - j3)I_3$; $0 = (-4 + j3)I_3 + (8 + j5)I_4$ (all units are in V, A, or Ω). Then use Ohm's law to calculate $v(t)$.
- $\mathbf{v(t)} = 145.52 \text{ V} \cos(2\pi 30t + 14.04\pi/180) -$ There are many different ways to label and solve this question using KCL and KVL. One possible set of equations is $I_2 - I_3 - I_4 = 0$; $I_1 + I_4 - I_5 = 0$;

$100 = 5I_2 - j10I_3$; $0 = j10I_3 + 10I_4 + j10I_5$; $0 = -j5I_1 - 10I_4 - 5I_2$ (all units are in V, A, or Ω). Then use Ohm's law to calculate $v(t)$.

10. $\mathbf{v(t) = 3.12 V \cos(2\pi 200t - 8.00\pi/180)}$ – There are many different ways to label and solve this question using KCL and KVL. One possible set of equations is $I_1 - I_2 - I_3 = 0$; $I_3 - 0.5(5I_2) - I_4 - I_5 = 0$; $5 = 2I_1 + 5I_2$; $0 = -5I_2 - j19.89I_3 + 7I_4$; $0 = -7I_4 - j7.96I_5$ (all units are in V, A, or Ω). Then use Ohm's law to calculate $v(t)$.

Complex Mesh Analysis

11. $\mathbf{v(t) = 50.46 V \cos(2\pi 50t + 17.91\pi/180)}$ – There are many different ways to label and solve this question using mesh analysis. One possible set of equations is $12I_1 + 24(I_1 - I_2) = 105$; $24(I_2 - I_1) + (9 + j12)I_2 = 0$ (all units are in V, A, or Ω). Then use Ohm's law to calculate $v(t)$.
12. $\mathbf{i(t) = 1.83 A \cos(2\pi 100t + 103.18\pi/180)}$ – There are many different ways to label and solve this question using mesh analysis. One possible set of equations is $15(I_1 - I_3) - j9.95(I_1 - I_2) = 21.21 + j21.21$; $-j9.95(I_2 - I_1) + j5.03(I_2 - I_3) + 20I_2 = 0$; $(30 - j6.37)I_3 + j5.03(I_3 - I_2) + 15(I_3 - I_1) = 0$ (all units are in V, A, or Ω).
13. $\mathbf{v(t) = 5.05 V \cos(2\pi 200t + 32.40\pi/180)}$ – There are many different ways to label and solve this question using mesh analysis. One possible set of equations is $130I_1 - j159.15(I_1 - I_2) + 80(I_1 - I_3) = 12$; $-j159.15(I_2 - I_1) + 56I_2 + 210(I_2 - I_3) = 0$; $80(I_3 - I_1) + 210(I_3 - I_2) + j119.38I_3 = 0$ (all units are in V, A, or Ω). Then use Ohm's law to calculate $v(t)$.
14. $\mathbf{v(t) = 8.09 V \cos(2\pi 5000t + 14.35\pi/180)}$ – There are many different ways to label and solve this question using mesh analysis. One possible set of equations is $20I_1 + 50(I_1 - I_2) = 6.43 - j7.66$; $50(I_2 - I_1) - j22.74I_2 + 20(I_2 - I_3) + 2V$; $V - 50I_1 + 50I_2 = 0$; $-2V + 20(I_3 - I_2) + 60I_3 = 0$ (all units are in V, A, or Ω). Then use Ohm's law to calculate $v(t)$.
15. $\mathbf{v(t) = 15.84 V \cos(2\pi 600t + 122.13\pi/180)}$ – There are many different ways to label and solve this question using mesh analysis. One possible set of equations is $I_1 - 6I_3 - I_2 = 0$; $30(I_1 - I_3) + 60(I_2 - I_3) - j40.19I_2 = 13.59 + j6.34$; $20I_3 - j50.05I_3 + 60(I_3 - I_2) + 30(I_3 - I_1) = 0$ (all units are in V, A, or Ω). Then use Ohm's law to calculate $v(t)$.

Complex Superposition

16. $\mathbf{v(t) = 13.06 V \cos(2\pi 10000t + 113.30\pi/180)}$ – The voltage contributed by the 5 V source is $3.84 V \cos(2\pi 10000t + 54.81\pi/180)$. The voltage contributed by the 30 V source is $11.52 V \cos(2\pi 10000t + 129.81\pi/180)$.

17. $\mathbf{v(t) = 31.62\ V \cos(2\pi 40t - 71.57\pi/180)}$ – The voltage contributed by the current source is $28.28\ V \cos(2\pi 40t - 45\pi/180)$. The voltage contributed by the voltage source is $14.14\ V \cos(2\pi 40t - 135\pi/180)$.
18. $\mathbf{v(t) = 417.23\ V \cos(2\pi 100t - 38.05\pi/180)}$ – The voltage contributed by the current source is $424.74\ V \cos(2\pi 100t - 42.27\pi/180)$. The voltage contributed by the voltage source is $31.94\ V \cos(2\pi 100t + 63.44\pi/180)$.
19. $\mathbf{i(t) = 2.23\ A \cos(2\pi 2000t + 64.57\pi/180)}$ – The current contributed by the current source is $2.23\ A \cos(2\pi 2000t - 25.73\pi/180)$. The current contributed by the voltage source is $3.16\ A \cos(2\pi 2000t + 109.42\pi/180)$.
20. $\mathbf{i(t) = 151.81\ mA \cos(2\pi 20t - 8.72\pi/180)}$ – The current contributed by the five volt voltage source is $19.42 \cos(2\pi 20t - 37.51\pi/180)$ mA. The current contributed by the ten volt voltage source is $21.97\ mA \cos(2\pi 20t + 30.62\pi/180)$. The current contributed by the current source is $117.89\ mA \cos(2\pi 20t - 10.95\pi/180)$.

Complex Source Transformation

21. $\mathbf{v(t) = 13.06\ V \cos(2\pi 10000t + 113.30\pi/180)}$ – Convert both sources to current sources. Then, combine all sources and impedances. Use Ohm's law to calculate the voltage. (This answer should be and is identical to question 16 in the complex superposition section.)
22. $\mathbf{v(t) = 31.62\ V \cos(2\pi 40t - 71.57\pi/180)}$ – Convert the voltage source to a current source. Then, combine all sources and impedances. Use Ohm's law to calculate the voltage. (This answer should be and is identical to question 17 in the complex superposition section.)
23. $\mathbf{i(t) = 294\ mA \cos(2\pi 10t - 11.84\pi/180)}$ – Apply repeated applications of source transformation until there is one current source in parallel with the $11\ \Omega$ resistor and one or more additional impedances. Then, apply the current divider rule to calculate the current through the resistor.
24. $\mathbf{v(t) = 29.81\ V \cos(2\pi 50000t + 3.86\pi/180)}$ – Apply repeated applications of source transformation until there is one voltage source in series with each passive element. Then, apply the voltage divider rule to calculate the voltage dropped over the capacitor.
25. $\mathbf{v(t) = 576.06\ mV \cos(2\pi 100t - 53.88\pi/180)}$ – Apply repeated applications of source transformation until there is one current source in series with one or more impedances. Then, use complex Ohm's law to calculate the voltage dropped over the $1.6\ mF$ capacitor.

Complex Thévenin and Norton's Theorems

26. The Thévenin equivalent circuit is shown in figure SOL.10 – Calculate the open-circuit voltage using a voltage divider, then deactivate the voltage source to calculate the equivalent impedance between nodes a and b.

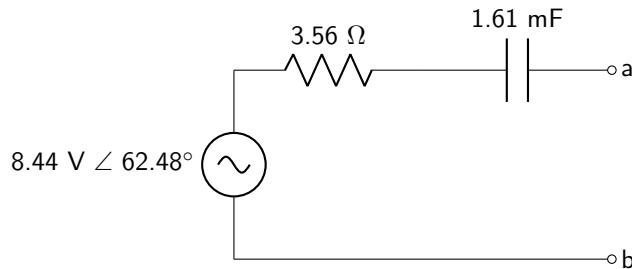


Figure SOL.10: Circuit diagram for complex Thévenin and Norton theorems answer 26.

27. The Thévenin equivalent circuit is shown in figure SOL.11 – Calculate the open-circuit voltage using any circuit analysis technique, then deactivate the current source to calculate the equivalent impedance between nodes a and b.

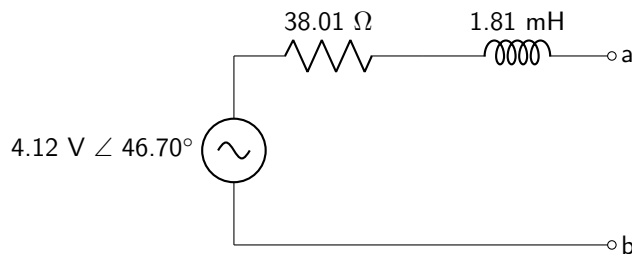


Figure SOL.11: Circuit diagram for complex Thévenin and Norton theorems answer 27.

28. The Thévenin equivalent circuit is shown in figure SOL.12 – Calculate the open-circuit voltage using any circuit analysis technique, then deactivate the voltage source to calculate the equivalent impedance between nodes a and b.

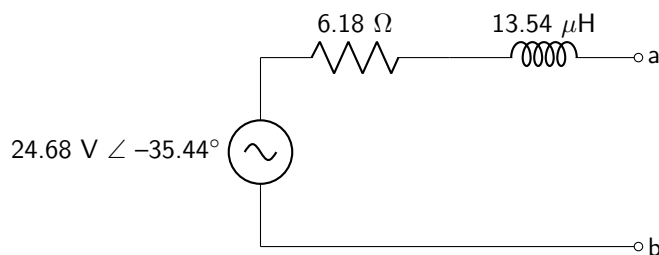


Figure SOL.12: Circuit diagram for complex Thévenin and Norton theorems answer 28.

29. **The Norton equivalent circuit is shown in figure SOL.13** – Calculate the open-circuit voltage and short-circuit current using any circuit analysis techniques. Then divide \mathbf{V}/\mathbf{I} to calculate the Norton equivalent impedance.

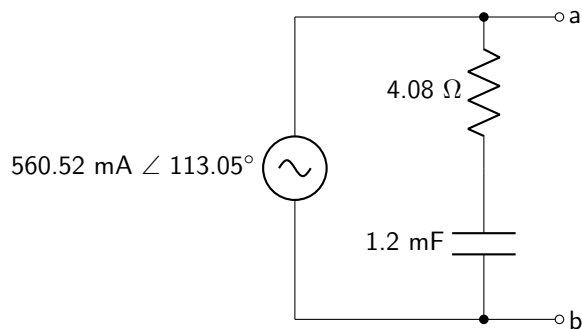


Figure SOL.13: Circuit diagram for complex Thévenin and Norton theorems answer 29.

30. **The Norton equivalent circuit is shown in figure SOL.14** – Calculate the open-circuit voltage and short-circuit current using any circuit analysis techniques. Then divide \mathbf{V}/\mathbf{I} to calculate the Norton equivalent impedance.

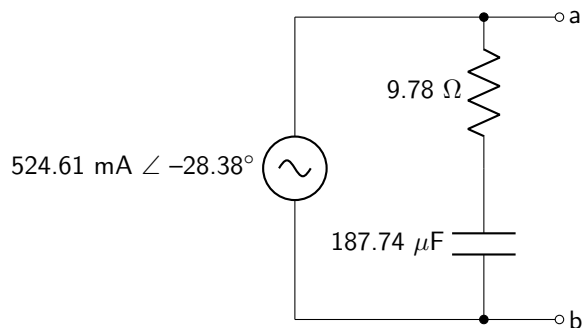


Figure SOL.14: Circuit diagram for complex Thévenin and Norton theorems answer 30.

Filters

31. **The filter is an LPF with center frequency of $1/\sqrt{LC}$, bandwidth of R/L , and quality factor of $\sqrt{L}/(R\sqrt{C})$** – Note that this is a series RLC circuit.
32. **The filter is an HPF with center frequency of $1/\sqrt{LC}$, bandwidth of $1/(RC)$, and quality factor of $R\sqrt{C}/\sqrt{L}$** – Note that this is a parallel RLC circuit.
33. **The filter is a BSF with center frequency of 100,000 rad/s, bandwidth of 100,000 rad/s, and quality factor of 1** – Note that this is a series RLC circuit.

34. **The filter is an HPF with cutoff frequency of $1/(R_1C)$** – Tau can be calculated from the first-order differential equation. The zero-input first-order differential equation is shown in equation SOL.6.

$$0 = \frac{dv_{out}}{dt} + \frac{1}{R_1C}v_{out} \quad (\text{SOL.6})$$

35. **The filter is an LPF with cutoff frequency of 10 rad/s** – Tau can be calculated from the first-order differential equation. The zero-input first-order differential equation (given that $R_1 = 30 \text{ k}\Omega$, $R_2 = 20 \text{ k}\Omega$, and $C = 5 \text{ }\mu\text{F}$) is shown in equation SOL.7.

$$0 = \frac{dv_{out}}{dt} + \frac{1}{R_2C}v_{out} \quad (\text{SOL.7})$$

Transfer Functions

36. **The transfer function is given in equation SOL.8, and the filter is an LPF** – Combine impedances R_2 and $1/(j\omega C)$ in parallel, then use a voltage divider to calculate the transfer function.

$$H(j\omega) = \frac{\frac{1}{CR_1}}{j\omega + \frac{R_1+R_2}{CR_1R_2}} \quad (\text{SOL.8})$$

37. **The transfer function is given in equation SOL.9, and the filter is a BPF** – Use a voltage divider to calculate the transfer function.

$$H(j\omega) = \frac{j\omega \frac{R}{L}}{-\omega^2 + j\omega \frac{R}{L} + \frac{1}{LC}} \quad (\text{SOL.9})$$

38. **The transfer function is given in equation SOL.10, and the filter is an HPF** – Combine impedances R and $1/(j\omega C)$ in parallel. Then use a voltage divider to calculate the transfer function.

$$H(j\omega) = \frac{-\omega^2 + j\omega \frac{1}{RC}}{-\omega^2 + j\omega \frac{1}{RC} + \frac{1}{LC}} \quad (\text{SOL.10})$$

39. **The transfer function is given in equation SOL.11, and the filter is an HPF** – Combine impedances $R_2 + j\omega L$ and R_1 in parallel. Then use a voltage divider (twice) to calculate the transfer function. Alternatively, use any other circuit analysis techniques to calculate the transfer function.

$$H(j\omega) = \frac{-\omega^2}{-\omega^2 + j\omega \left(\frac{1}{R_1C} + \frac{R_2}{L} \right) + \frac{R_1+R_2}{R_1LC}} \quad (\text{SOL.11})$$

40. **The transfer function is given in equation SOL.12, and the filter is an LPF** – Combine impedances R_3 and $1/(j\omega C_2)$ in parallel. Then use a voltage divider at the non-inverting node to calculate the op-amp input node voltage. Then, use the ideal op-amp properties to calculate the

transfer function.

$$H(j\omega) = \frac{j\omega \left(\frac{1}{C_1 R_1} \right) + \frac{R_2 + R_3}{C_1 C_2 R_1 R_2 R_3}}{-\omega^2 + j\omega \left(\frac{1}{C_2 R_3} + \frac{1}{C_1 R_1} \right) + \frac{1}{C_1 C_2 R_1 R_3}} \quad (\text{SOL.12})$$

Chapter 10

Phase and Root Mean Square

1. $|\mathbf{V}_{\text{RMS}}| = 84.85 \text{ VRMS}$, $|\mathbf{I}_{\text{RMS}}| = 3.54 \text{ ARMS}$, $\theta = 60^\circ$ – Both signals are sinusoids, so use equation 10.4 to calculate the RMS values. Use equation 10.1 to calculate the phase difference.
2. $|\mathbf{V}_{\text{RMS}}| = 106.07 \text{ VRMS}$, $|\mathbf{I}_{\text{RMS}}| = 2.12 \text{ ARMS}$, $\theta = 25^\circ$ – Both signals are sinusoids, so use equation 10.4 to calculate the RMS values. Use equation 10.1 to calculate the phase difference.
3. $|\mathbf{V}_{\text{RMS}}| = 155.56 \text{ VRMS}$, $|\mathbf{I}_{\text{RMS}}| = 4.24 \text{ ARMS}$, $\theta = 160^\circ$ – Both signals are sinusoids, so use equation 10.4 to calculate the RMS values. Use equation 10.1 to calculate the phase difference.
4. $|\mathbf{V}_{\text{RMS}}| = 42.43 \text{ VRMS}$, $|\mathbf{I}_{\text{RMS}}| = 4.50 \text{ ARMS}$, $\theta = -32.01^\circ$ – Use complex Ohm's law to calculate the current flowing through the circuit. Both signals are sinusoids, so use equation 10.4 to calculate the RMS values. Use equation 10.1 to calculate the phase difference.
5. $|\mathbf{V}_{\text{RMS}}| = 70.71 \text{ VRMS}$, $|\mathbf{I}_{\text{RMS}}| = 7.07 \text{ ARMS}$, $\theta = 53.13^\circ$ – Use complex Ohm's law to calculate the current flowing through the circuit. Both signals are sinusoids, so use equation 10.4 to calculate the RMS values. Use equation 10.1 to calculate the phase difference.

Complex Power

6. $\mathbf{S}_{\text{LOAD}} = 3453.00 - j2555.89 \text{ VA}$, $\text{pf} = 0.804$ – The current through the load is given. Find the equivalent impedance of the load and use equation 10.9 to calculate the power through the load. Then use equation 10.14 to calculate the power factor.
7. $\mathbf{S}_{\text{LOAD}} = 338.79 + j132.13 \text{ VA}$, $\text{pf} = 0.932$ – Find the equivalent impedance of the load, then use a complex voltage divider to calculate the voltage dropped over the load. Use equation 10.8 to calculate the power through the load. Then use equation 10.14 to calculate the power factor.
8. $\mathbf{S}_{\text{LOAD}} = 8.31 + j11.08 \text{ VA}$, $\text{pf} = 0.600$ – Find the equivalent impedance of the load, then use a complex voltage divider to calculate the voltage dropped over the load. Use equation 10.8 to calculate the power through the load. Then use equation 10.14 to calculate the power factor.
9. $\mathbf{S}_{\text{LOAD}} = 557.38 - j203.20 \text{ VA}$, $\text{pf} = 0.940$ – Find the equivalent impedance of the load, then use a complex voltage divider to calculate the voltage dropped over the load. Use equation 10.8 to calculate the power through the load. Then use equation 10.14 to calculate the power factor.

10. $S_{\text{LOAD}} = 631.25 + j1558.36 \text{ VA}$, $\text{pf} = 0.375$ – Find the equivalent impedance of the load, then use a complex voltage divider to calculate the voltage dropped over the load. Use equation 10.8 to calculate the power through the load. Then use equation 10.14 to calculate the power factor.

Maximum Power Transfer

11. **The load should consist of a 205 Ω resistor in series with a 48.8 μF capacitor, the power consumed by the load will be 76.2 W** – Calculate the Thévenin equivalent impedance. The load impedance should be equal to the complex conjugate of that value. Use equation 10.16 to calculate the maximum power consumption.
12. **The load should consist of a 5.9 Ω resistor in series with a 263 μF capacitor, the power consumed by the load will be 62.52 W** – Calculate the Thévenin equivalent impedance. The load impedance should be equal to the complex conjugate of that value. Use equation 10.16 to calculate the maximum power consumption.
13. **The load should consist of a 80 Ω resistor in series with a 13 mH inductor, the power consumed by the load will be 12.9 mW** – Calculate the Thévenin equivalent impedance. The load impedance should be equal to the complex conjugate of that value. Use equation 10.16 to calculate the maximum power consumption.
14. **The load should consist of a 11.2 Ω resistor in series with a 32.9 mH inductor, the power consumed by the load will be 142.7 W** – Calculate the Thévenin equivalent impedance. The load impedance should be equal to the complex conjugate of that value. Use equation 10.16 to calculate the maximum power consumption.
15. **The load should consist of a 6.4 Ω resistor in series with a 657 μH inductor, the power consumed by the load will be 553.4 W** – Calculate the Thévenin equivalent impedance. The load impedance should be equal to the complex conjugate of that value. Use equation 10.16 to calculate the maximum power consumption.

Power Factor Correction

16. **The initial power factor is 0.692, and to increase the power factor to 0.800, a 7.9 μF capacitor must be placed in parallel with the load** – Start by calculating the load impedance and the voltage dropped over the load. Then use equation 10.8 to calculate the power consumed by the load. The initial power factor can be calculated with equation 10.14. Use equations 10.17 and 10.18 to determine the value of the corrective capacitor.
17. **The initial power factor is 0.221, and to increase the power factor to 0.860, a 85.4 mH inductor must be placed in parallel with the load** – Start by calculating the load impedance and

the voltage dropped over the load. Then use equation 10.8 to calculate the power consumed by the load. The initial power factor can be calculated with equation 10.14. Use equations 10.19 and 10.20 to determine the value of the corrective inductor.

18. **The initial power factor is 0.951, and to increase the power factor to 0.990, a 5.9 μF capacitor must be placed in parallel with the load** – Start by calculating the load impedance and the voltage dropped over the load. Then use equation 10.8 to calculate the power consumed by the load. The initial power factor can be calculated with equation 10.14. Use equations 10.17 and 10.18 to determine the value of the corrective capacitor.
19. **The initial power factor is 0.922, and to increase the power factor to 0.980, a 1.3 H inductor must be placed in parallel with the load** – Start by calculating the load impedance and the voltage dropped over the load. Then use equation 10.8 to calculate the power consumed by the load. The initial power factor can be calculated with equation 10.14. Use equations 10.19 and 10.20 to determine the value of the corrective inductor.
20. **The initial power factor is 0.921, and to increase the power factor to 0.960, a 8.6 μF capacitor must be placed in parallel with the load** – Start by calculating the load impedance and the voltage dropped over the load. Then use equation 10.8 to calculate the power consumed by the load. The initial power factor can be calculated with equation 10.14. Use equations 10.17 and 10.18 to determine the value of the corrective capacitor.

Chapter 11

Laplace Transforms

1. The Laplace transform of $f(t)$ is given in equation SOL.13.

$$F(s) = \frac{2}{s+5} \quad (\text{SOL.13})$$

2. The Laplace transform of $f(t)$ is given in equation SOL.14.

$$F(s) = \frac{7s^2 + 54s + 80}{s(s^2 + 8s + 16)} \quad (\text{SOL.14})$$

3. The Laplace transform of $f(t)$ is given in equation SOL.15.

$$F(s) = \frac{2s + 42}{s^2 + 2s + 101} \quad (\text{SOL.15})$$

4. The Laplace transform of $v(t)$ is given in equation SOL.16.

$$V(s) = \frac{3}{s + 0.25} \quad (\text{SOL.16})$$

5. The Laplace transform of $i(t)$ is given in equation SOL.17.

$$I(s) = \frac{10s + 55}{s^2 + 6s + 109} \quad (\text{SOL.17})$$

Inverse Laplace Transforms

6. The function $f(t)$ is given in equation SOL.18. The single term contributes to the transient response of the circuit.

$$f(t) = 10 e^{-10t} u(t) \quad (\text{SOL.18})$$

7. The partial fraction expansion of $F(s)$ is given in equation SOL.19, and the function $f(t)$ is given in equation SOL.20. The term 0.4 contributes to the steady-state response. The terms $-52 t e^{-5t}$ and $9.6 e^{-5t}$ contribute to the transient response.

$$F(s) = \frac{0.4}{s} + \frac{-52}{(s+5)^2} + \frac{9.6}{(s+5)} \quad (\text{SOL.19})$$

$$f(t) = [0.4 - 52 t e^{-5t} + 9.6 e^{-5t}] u(t) \quad (\text{SOL.20})$$

8. The partial fraction expansion of $F(s)$ is given in equation SOL.21, and the function $f(t)$ is given in equation SOL.22. The terms $0.11 \cos(4t)$ and $0.066 \sin(4t)$ contribute to the steady-state response. The terms $0.098 e^{-t}$ and $-0.208 e^{-4t}$ contribute to the transient response.

$$F(s) = \frac{0.055 + j0.033}{(s+j4)} + \frac{0.055 - j0.033}{(s-j4)} + \frac{0.098}{(s+1)} + \frac{-0.208}{(s+4)} \quad (\text{SOL.21})$$

$$f(t) = [0.11 \cos(4t) + 0.066 \sin(4t) + 0.098 e^{-t} - 0.208 e^{-4t}] u(t) \quad (\text{SOL.22})$$

9. The partial fraction expansion of $F(s)$ is given in equation SOL.23, and the function $f(t)$ is given in equation SOL.24. All of the terms in the function contribute to the transient response.

$$F(s) = \frac{0.052}{(s+2)^2} + \frac{-0.173}{(s+2)} + \frac{0.087 - j0.331}{(s+2.5-j7.6)} + \frac{0.087 + j0.331}{(s+2.5+j7.6)} \quad (\text{SOL.23})$$

$$f(t) = [0.052 t e^{-2t} - 0.173 e^{-2t} + e^{-2.5t} (0.173 \cos(7.6t) + 0.663 \sin(7.6t))] u(t) \quad (\text{SOL.24})$$

10. The partial fraction expansion of $F(s)$ is given in equation SOL.25, and the function $f(t)$ is given in

equation SOL.26. The terms 0.004 , $-0.021 \cos(5t)$, and $0.028 \sin(5t)$ contribute to the steady-state response. The terms $0.312 t e^{-10t}$ and $0.017 e^{-10t}$ contribute to the transient response.

$$F(s) = \frac{0.004}{s} + \frac{-0.011 - j0.014}{(s + j5)} + \frac{-0.011 + j0.014}{(s - j5)} + \frac{0.312}{(s + 10)^2} + \frac{0.017}{(s + 10)} \quad (\text{SOL.25})$$

$$f(t) = [0.004 - 0.021 \cos(5t) + 0.028 \sin(5t) + 0.312 t e^{-10t} + 0.017 e^{-10t}] u(t) \quad (\text{SOL.26})$$

s-Domain Analysis

11. The s-domain transformed circuit is shown in figure SOL.15. $V(s)$ is given in symbolic form in equation SOL.27 and $v(t)$ is given in equation SOL.28.

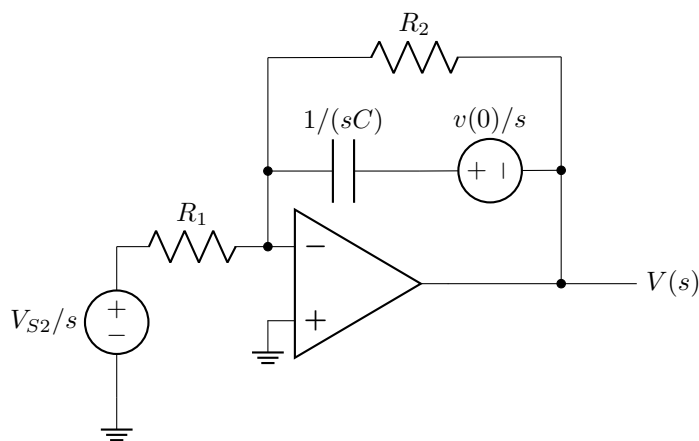


Figure SOL.15: Circuit diagram for s-domain analysis answer 11.

$$V(s) = \frac{sv(0) - \frac{V_{S2}}{CR_1}}{s(s + \frac{1}{CR_2})} \quad (\text{SOL.27})$$

$$v(t) = [-4 \text{ V} - 2 \text{ V} e^{-40t}] u(t) \quad (\text{SOL.28})$$

12. The s-domain transformed circuit is shown in figure SOL.16. $V(s)$ is given in symbolic form in equation SOL.29.

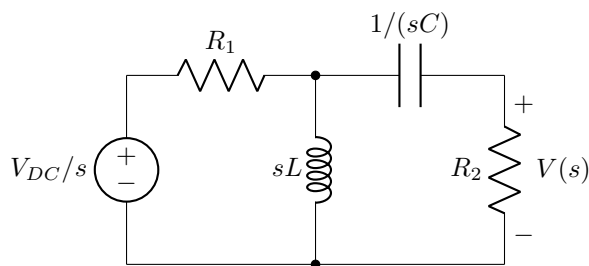


Figure SOL.16: Circuit diagram for s-domain analysis answer 12.

$$V(s) = \frac{\left[\frac{V_{DC} R_2}{R_1 + R_2} \right] s}{s^2 + \left[\frac{L + C R_1 R_2}{LC(R_1 + R_2)} \right] s + \left[\frac{R_1}{LC(R_1 + R_2)} \right]} \quad (\text{SOL.29})$$

13. The s-domain transformed circuit is shown in figure SOL.17. $V(s)$ is given in symbolic form in equation SOL.30 and $v(t)$ is given in equation SOL.31.

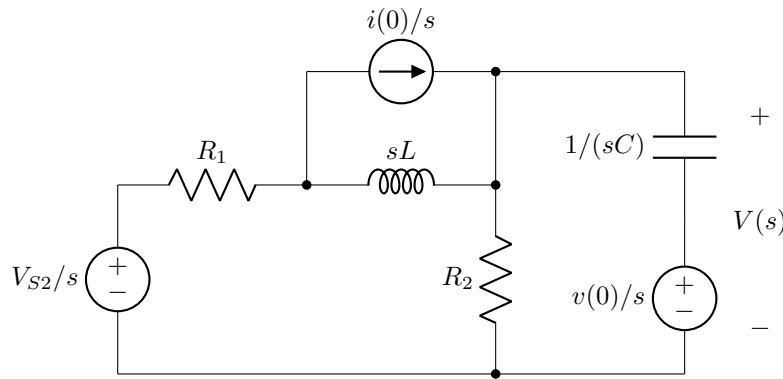


Figure SOL.17: Circuit diagram for s-domain analysis answer 13.

$$V(s) = \frac{\left[\frac{V_{S1} R_2}{R_1 + R_2} \right] s^2 + \left[\frac{L V_{S1} + C V_{S1} R_1 R_2}{LC(R_1 + R_2)} \right] s + \left[\frac{V_{S2}}{LC} \right]}{s \left(s^2 + \left[\frac{1}{R_2 C} + \frac{R_1}{L} \right] s + \left[\frac{R_1 + R_2}{LC R_2} \right] \right)} \quad (\text{SOL.30})$$

$$v(t) = \left[6.67 \text{ V} + e^{-7000t} (-4.67 \text{ V} \cos(3316.63t) + 9.85 \text{ V} \sin(3316.63t)) \right] u(t) \quad (\text{SOL.31})$$

14. The s-domain transformed circuit is shown in figure SOL.18. $I(s)$ is given in symbolic form in equation SOL.32 and $i(t)$ is given in equation SOL.33.

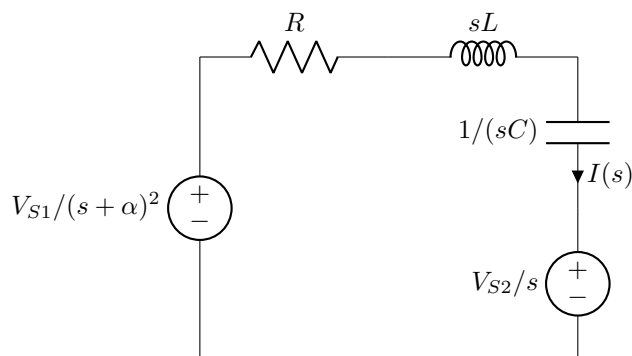


Figure SOL.18: Circuit diagram for s-domain analysis answer 14.

$$I(s) = \frac{\left[\frac{-V_{S2}}{L}\right] s^2 + \left[\frac{V_{S1} - 2\alpha V_{S2}}{L}\right] s + \left[\frac{-V_{S2}\alpha^2}{L}\right]}{(s + \alpha)^2 \left(s^2 + \left[\frac{R}{L}\right] s + \left[\frac{1}{LC}\right]\right)} \quad (\text{SOL.32})$$

$$i(t) = [-1.6 \text{ mA/s } t e^{-4t} + 0.4 \text{ mA } e^{-4t} + e^{-62.5t} (-0.4 \text{ mA } \cos(496.08t) + 50.44 \text{ mA } \sin(496.08t))] u(t) \quad (\text{SOL.33})$$

15. The s-domain transformed circuit is shown in figure SOL.19. $I(s)$ is given in symbolic form in equation SOL.34 and $i(t)$ is given in equation SOL.35.

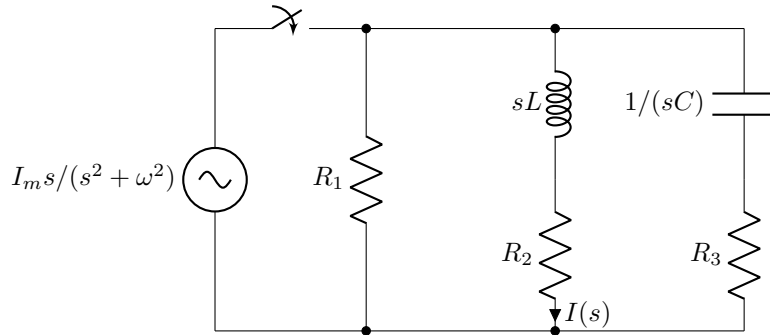


Figure SOL.19: Circuit diagram for s-domain analysis answer 15.

$$I(s) = \frac{\left[\frac{CR_1R_3I_m}{LC(R_1+R_3)}\right] s^2 + \left[\frac{R_1I_m}{LC(R_1+R_3)}\right] s}{(s^2 + \omega^2) \left(s^2 + \left[\frac{1}{C(R_1+R_3)} + \frac{R_2R_3+R_1R_3+R_1R_2}{L(R_1+R_3)}\right] s + \left[\frac{R_1+R_2}{LC(R_1+R_3)}\right]\right)} \quad (\text{SOL.34})$$

$$i(t) = [0.5 \text{ A } \cos(60t) - 0.01 \text{ A } \sin(60t) + e^{-154.33t} (-0.5 \text{ A } \cos(990.5t) + 0.07 \text{ A } \sin(990.5t))] u(t) \quad (\text{SOL.35})$$

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